REAL ANALYSIS PRELIM EXAMINATION

SPRING 1998

Complete as many as possible of the questions below. A smaller set of complete solutions may carry more weight than a larger set of partial solutions.

Unless otherwise stated, "measurable" will mean measurable with respect to ordinary Lebesgue measure on R^n.

1. Let f be a real-valued continuous function on a closed interval $[a,b]$. Prove that f is *uniformly continuous* on $[a,b]$.

2. Let $f_n(x) = \frac{nx^2}{1+nx^2}$. Determine whether the sequence $\{f_n\}$ converges uniformly on each of the intervals $[0,1]$ and $[1,\infty)$. Justify your results.

3. Let $f(x)$ be a real-valued C^2 function on a closed interval $[a,b]$. Given $\varepsilon > 0$, prove there exists a polynomial $p(x)$ which satisfies *both* of the following conditions:
$$\sup_{x\in[a,b]} |f'(x) - p'(x)| < \varepsilon$$
$$\sup_{x\in[a,b]} |f(x) - p(x)| < \varepsilon$$

4. Prove the Riesz-Fisher Theorem: Let $\{\phi_k\}$ be any orthonormal system in $L^2(E)$ where E is a fixed measurable subset of $\mathbf{R}^n$. Let $\{c_k\}$ be any sequence in l^2. Prove there exists an f in $L^2(E)$ such that $S[f] = \sum c_k\phi_k$ -i.e. such that $\{c_k\}$ is the sequence of Fourier coefficients of f with respect to $\{\phi_k\}$. In addition prove that Parseval's formula holds:
$$\sum_k |c_k|^2 = \int_E |f(x)|^2 dx$$

5. The convolution of two functions f and g which are measurable in $\mathbf{R}^n$ is defined by
$$(f*g)(x) = \int_{R^n} f(t)g(x-t)dt$$
Prove that if $1 < p < \infty$, $f \in L^p(R^n)$, $g \in L^1(R^n)$ then $f*g \in L^p(R^n)$.

6. Let $\{f_n\}$ be a sequence of functions in $L^1(R^n)$ such that $\|f - f_n\|_{L^1} \to 0$. Let $\hat{f}(\omega)$ denote the Fourier transform of f. Suppose $\left|\hat{f}_n(\omega)\right| < \left(\frac{1}{1+|\omega|}\right)^{-(\frac{1}{2}+\varepsilon)}$ for some $\varepsilon > 0$. Prove
   a. $\hat{f}_n(\omega) \to \hat{f}(\omega)$ pointwise as $n \to \infty$
   b. $\|f_n - f\|_{L^2(R^n)} \to 0$ as $n \to \infty$

7. Give an example of a function $f(x,y)$ defined on $I = [0,1]\times[0,1]$ for which each of the

iterated integrals $\int_0^1\left(\int_0^1 f(x,y)dx\right)dy$ and $\int_0^1\left(\int_0^1 f(x,y)dy\right)dx$ is finite, but $\int\int_I |f(x,y)|dxdy$ is infinite.

8. Suppose $f_k \to f$ in $L^p(R)$, $1 \le p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty \le M$ for all k. Prove that $f_k g_k \to fg$ in $L^p(R)$.

9. Consider the sequence $f_n(x) = \cos(nx)$ on $[0, 2\pi]$.
   a. Prove that $f_n \to 0$ weakly in $L^2([0, 2\pi])$
   b. Prove that f_n does not converge to 0 strongly in $L^2([0, 2\pi])$.
   c. Prove that f_n does not converge to 0 in measure.

10. A real valued function f on R is Lipschitz if $|f(x) - f(y)| \le C|x - y|$ for any $x,y \in R$ where C is a fixed constant.
   a. Prove that a subset E of R is measurable if and only if $E = H \cup Z$ where H is an F_σ set and Z is of measure 0. (An F_σ set is a countable union of closed sets.)
   b. Prove that the image of a measurable set under a Lipschitz function f is measurable.