REAL

Real Analysis Preliminary Examination
Spring, 1997

DO ANY 9 PROBLEMS.

<u>Note:</u> Throughout the exam, $L^1(R)$ and $L^2(R)$ refer, respectively, to the space of integrable and square integrable functions on the real line, *with respect to Lebesgue measure.*

1. (a) Prove that the uniform limit of a sequence of continuous real-valued functions on $[0,1]$ is continuous.

   (b) Give an example of a sequence f_n of continuous real-valued functions on $[0,1]$ such that f_n converges pointwise to a discontinuous function f on $[0,1]$.

   (c) Give an example of a sequence f_n of continuous real-valued functions on $[0,1]$ such that f_n converges pointwise, but not uniformly, to a continuous f on $[0,1]$.

2. Let $f : R^2 \to R^2$

   (a) Define "f is differentiable at $p_o = (x_o, y_o)$"

   (b) Prove that if f is differentiable at p_o, then the partial derivative $\left(\frac{\partial f}{\partial x}\right)_{p_o}$ exists.

   (c) Give an example of $f : U \to R^2$, where U is a non-empty open subset of R^2, such that f is continuously differentiable on U, with non-vanishing Jacobian, but f is not 1-1 on U.

3. Assume the functions $f(t)$ and $tf(t)$ are in $L^1(R)$. Let $g(x) = \int_{-\infty}^{\infty} f(t)e^{ixt}\,dt$. Show that $g'(x)$ exists, and that $g'(x) = \int_{-\infty}^{\infty} f(t)ite^{ixt}\,dt$.

4. For $A, B \subset R$, let $A + B = \{a + b | a \in A, b \in B\}$. Let I_n be the open interval $(-\frac{1}{n}, \frac{1}{n})$. Denote Lebesgue measure on R by m. Prove that if C is a compact subset of R, with $m(C) > 0$, then

   (a) for each $n, C + I_n$ is measurable

   (b) for some n, $m(C + I_n) < 2m(C)$.

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5. Let $f \in L^1(R)$, and for $y \in R$, let $f_y(x) = f(x-y)$. Prove that $\lim_{y\to 0} \|f_y - f\|_1 = 0$.

6. Let μ and ν be finite, positive measures on a measurable space (X, β).

   (a) Show that μ is absolutely continuous with respect to $\mu + \nu$.

   (b) Explain how to find the Radon-Nikodym derivative of μ with respect to $\mu + \nu$.

7. Let $f, g \in L^1(R)$. Give a careful argument that the integral

$$\int_R f(t)g(x-t)\,dt$$

exists for almost all x, and defines an integrable function of x on R (with respect to Lebesgue measure).

8. Prove that a normed linear space $X$is a Banach space if and only if, for each sequence x_n in X with $\Sigma_{n=1}^{\infty}\|x_n\| < \infty$, there exists $x \in X$ such that $x = \lim_{n\to\infty} \Sigma_{i=1}^{n} x_i$.

9. Let C be the space of all sequences $\{a_n\}$ of real numbers for which $\lim_{n\to\infty} a_n$ exists, and let l^∞ be the space of all bounded sequences, with the sup norm.

   (a) Prove that there exists $\Phi \in (l^\infty)'$ such that on C, $\Phi(\{a_n\}) = \lim_{n\to\infty} a_n$

   (b) Prove that Φ is not of the form $\Phi(\{a_n\}) = \Sigma_{i=1}^{\infty} a_n b_n$ for any sequence $b_n \in l^1$.

10. Let $f_n \in L^2(R)$ such that for each $g \in L^2(R)$, $\int f_n(x)g(x)\,dx$ converges, as $n \to \infty$. Prove that there exists $f \in L^2(R)$ such that $\int f_n(x)g(x)\,dx \to \int f(x)g(x)\,dx$, as $n \to \infty$, for all $g \in L^2(R)$.