Real Analysis Preliminary Exam, August 2^{th}, 1999

1. Prove that every real-valued continuous function on $[0, \pi]$ can be uniformly approximated with trigonometric polynomials of the form $a_0 + a_1\cos(x) + b_1\sin(x) + \ldots + a_n\cos(nx) + b_n\sin(nx)$, $a_i, b_i \in \mathbb{R}$. Also, prove the same fact for real-valued continuous functions f on $[-\pi, \pi]$ for which $f(-\pi) = f(\pi)$.

2. Let $S_n(x) := \sum_{k=1}^{n} \frac{\sin(kx)}{k^2}$ for $x \in [0, 2\pi]$, $n \geq 1$. Show that the following limit exists for every Lebesgue integrable function f on $[0, 2\pi]$:

$$\lim_{n\to\infty} \int_0^{2\pi} S_n(x) f(x)\,dx.$$

3. Let $\varphi(x) = 2x - x^2$, $x \in \mathbb{R}$. For every Lebesgue measurable set $E \subset \mathbb{R}$ let $\mu_\varphi(E) = \mu(\varphi^{-1}(E))$, where μ is the Lebesgue measure. Show that μ_φ is a measure which is absolutely continuous with respect to μ and compute the Radon-Nikodym derivative of μ_φ with respect to μ.

4. Let A and B be measurable subsets of the reals and let μ be the Lebesgue measure. Suppose that $0 < \mu(A), \mu(B) < \infty$. Prove the identity

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_B(t)\chi_{A+x}(t)\,d\mu(t)\,d\mu(x) = \mu(A)\mu(B),$$

and show that there exists an x_0 such that $\mu(B \cap (A + x_0)) > 0$.

5. Give an example of a map which is continuous at all the points of the Cantor set and discontinuous at all the other points in $[0,1]$.

6. If f is a differentiable mapping of a convex open set $E \subset \mathbb{R}^2$ $(n > 1)$ into $\mathbb{R}$, and $\frac{\partial f}{\partial x}(x,y) = 0$ for every $(x,y) \in E$, prove that f depends only of the variable y. Show that this is not true anymore if for instance $E = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$.

7. State the implicit function theorem. Prove this theorem for the case of a linear map.

8. Let M be the collection of continuous maps f on $[0,1]$ with the property:

$$\int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt = 1.$$

Show that M is a closed (with respect to the usual norm on continuous functions $\|f\|_\infty = \sup\{f(x) : \ x \in [0,1]\}$) and convex set, but there is no element in M of minimal norm (i.e. there exist no $f \in M$ such that $\|f\|_\infty = \inf\{\|g\|_\infty ; \ g \in M\}$).

9. Consider the continuous function $f : (0,\infty) \to \mathbb{R}$ with the property $\lim_{n\to\infty} f(nx) = 0$ for every $x \in [1,2]$. Use the Baire category theorem to show that $\lim_{x\to\infty} f(x) = 0$.