REAL ANALYSIS PRELIMINARY EXAM, FALL 1997

1. **a)** Show that the locus

$$2x + y + z + \sin z = 0$$

is the graph of a smooth function $z = f(x, y)$ in the neighborhood of the origin in $\mathbb{R}^3$.
b) Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0, 0)$.

2. Let $\mathcal{F}$ be a family of continuous nonnegative functions on a metric space X. Show that the nonnegative function

$$F(x) := \inf\{f(x) : f \in \mathcal{F}\}$$

is upper semicontinuous, i.e. $F^{-1}[0, b)$ is open for every $b > 0$.

3. Let $E = \{x_1, x_2, \ldots\} \subset [0, 1]$ be countably infinite. Let $a_1, a_2, \ldots > 0$ with $\sum_i a_i < \infty$.
a) Prove that

$$f(x) := \sum\{a_i : x_i \leq x\}$$

is right continuous everywhere in $[0, 1]$, and that the set of discontinuities of f is precisely E.
b) Prove that the range of f is a nowhere dense subset of $\mathbb{R}$.

4.a) Give an example of a sequence of nonnegative measurable functions $f_n : [0, 1] \to \mathbb{R}$ with $f_n \to 0$ pointwise but $\int_0^1 f_n \to 1$.
b) Is there a sequence of nonnegative measurable functions on $[0, 1]$ with $f_n \to 1$ pointwise and $\limsup \int_0^1 f_n < 1$? What if the nonnegativity condition is removed?

5. Prove that if $E \subset [0, 1]$ is a measurable set then

$$\lim_{n\to\infty} \int_E \sin^2 nx \, dx = \frac{1}{2} m(E).$$

(Hint: prove it first under the stronger assumption that E is open.)

6. Let $E \subset [0, 1]^2$ be Lebesgue measurable. Prove that there is a sequence $y_1, y_2, \ldots$ such that $F := \bigcup_i \{x \in \mathbb{R} : (x, y_i) \in E\} \times \mathbb{R}$ includes almost all of E, in the sense that $E - F$ has measure zero in $[0, 1]^2$.

7. Let X be a normed linear space (not necessarily complete) and X^* its dual space (i.e. the space of all bounded linear functionals on X).
a) Define the dual norm $|| \, ||_*$ on X^* and show that X^* is complete under this norm.
b) Suppose $v_1, v_2, \cdots \in X^*$ converge to $v \in X^*$ in the weak* topology. Show that $||v||_* \leq \liminf ||v_i||_*$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

8. **a)** State the closed graph theorem.
b) Give an example of a normed vector space X and a linear functional λ on X such that the graph of λ is a closed subspace of $X \oplus \mathbb{R}$, but λ is not continuous.

9. Let (X, μ) be a σ-finite measure space with $\mu(X) = \infty$. Show that, for every $p \in [0, \infty]$, there is an isometry of l^p into $L^p(X, \mu)$.