Real

Real Analysis Prelim

1. a) Let $f(x)$ be a bounded piecewise continuous function on $[0, 1]$. Prove

$$e^{\int_0^1 f(x)dx} \leq \int_0^1 e^{f(x)}\, dx$$

b) Let $\{a_1, \ldots, a_n\}$ be a set of positive numbers. Prove

$$\frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots a_n}{n}.$$

2. For Y a set and $\mathcal{A} \subset P(Y)$ ($P(Y)$ = set of subsets of Y) let $\mathcal{B}(\mathcal{A})$ be the sigma algebra generated by $\mathcal{A}$. For $i = 1, 2,$ let X_i be a set and $\mathcal{B}_i$ a sigma algebra on X_i. Suppose $\mathcal{A} \subset \mathcal{B}_2$ and $\mathcal{B}(\mathcal{A}) = \mathcal{B}_2$. Prove the following: A function $f: X_1 \to X_2$ is measureable if and only if $f^{-1}(\mathcal{A}) \subset \mathcal{B}_1$

3. Suppose $f(x)$ is continous even and $f \in \mathcal{L}^1(\mathbb{R})$.

Prove $\lim_{\epsilon \to 0^+} \frac{2}{\pi} \int_0^\infty e^{-\epsilon u} \cos xu \left(\int_0^\infty f(t) \cos tu\, dt \right) du = f(x).$

4. For $\{f_n: n \geq 1\}$ a sequence in $\mathcal{L}^2(X, \mu)$ and $f \in \mathcal{L}^2(X, \mu)$, prove the following: The sequence converges $\{f_n: n \geq 1\}$ converges to f in mean if and only if a) $\lim_{n\to\infty} (f_n, g) = (f, g)$ for any $g \in \mathcal{L}^2(X, \mu)$ and b) $\lim_{n\to\infty} \|f_n\|_2 = \|f\|_2$.

5. Let $\{f_n: n \geq 1\}$ be a sequence of continuous functions on the open interval (a, b) with $a < b$. Suppose $\sup_n f_n(c) < \infty$ for any $a < c < b$. Prove there exist α and β with $a < \alpha < \beta < b$ such that

$$\sup_{\alpha < x < \beta} \left(\sup_n f_n(x) \right) < \infty.$$

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6. Consider the transformation $T: (x, y, z) \to (u, v, w)$ where $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $w = x^3 + y^3 + z^3$.

   a) Prove T maps some neighborhood of $(-1, 0, 1)$ one to one onto a neighborhood of $(0, 2, 0)$.

   b) Does T map a neighborhood of $(0, 0, 2)$ one to one onto a neighborhood of $(2, 4, 8)$?

7. Let S be a subspace of $\mathcal{L}^2[0, 1]$, and suppose this is a constant K such that $|f(x)| \leq K\|f\|_2$ for all $f \in S$ and almost all $x \in [0, 1]$. Prove $\dim S \leq K^2$.

8. a) Let $q(x)$ and $p(x)$ be continuous functions on $[a, b]$ with $p(x)$ positive and monotonically decreasing. Prove

$$\int_a^b p(x)\, q(x)\, dx = p(a) \int_a^c q(x)\, dx$$

   for some $a \leq c \leq b$.

   b) Prove $\cancel{\int_0^\infty \frac{\sin x}{x}\, dx =} \lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\, dx$ exists.

9. Suppose f is Lebesgue integrable on $[a, b]$ and F is defined by

$$F(x) = \int_a^x f(t)\, dt.$$

   Prove F is continuous and of bounded variation on $[a, b]$.

10. If $F(x, y) = \int_x^{x^2 + y^2} e^{-yt^2 - xt}\, dt$ evaluate $F_x(x, y)$ and $F_y(x, y)$.