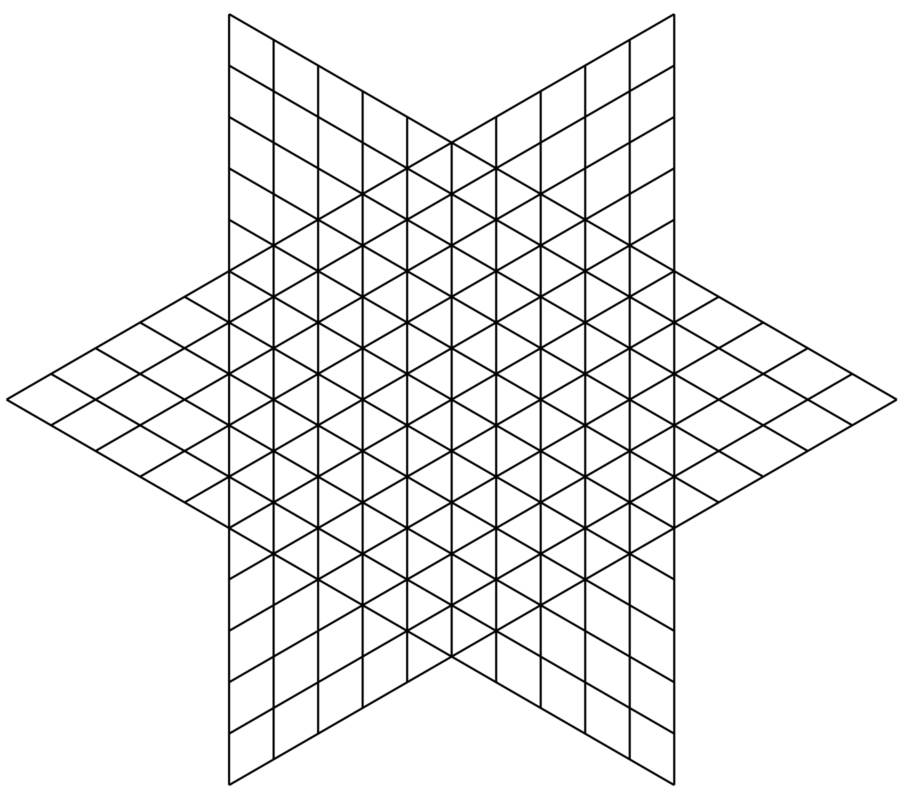
Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
October 20, 2018

WITH SOLUTIONS

Problem 1 (Triangles and tribulations)**.** How many triangles are in the figure below? Only count those triangles whose edges lie on the lines shown.



Answer. 1240 (triangles)

Solution. Every triangle in the figure is uniquely determined by choosing the three lines on which its edges lie. On the other hand, every choice of three lines (with one line in each of the three possible directions) gives rise to a triangle except if all three lines happen to intersect in a point. There are 11 lines in each direction, so this gives $11^3 = 1331$ possible choices of lines. The number of times the three lines can intersect in a point is exactly the count of points inside or on the central hexagon. We can count these either manually, or by say starting at the central point and counting the points on concentric hexagons (there are 6 points on the hexagon around the center, 12 points on the hexagon around that, etc.). This leads to a count of $1 + 6(1 + 2 + 3 + 4 + 5) = 91$ triple intersection points, giving a final count of

$$11^3 - 91 = 1240$$

triangles.

Problem 2 (Circular reasoning)**.** Figures 1–3 below are the first three in a sequence of figures.

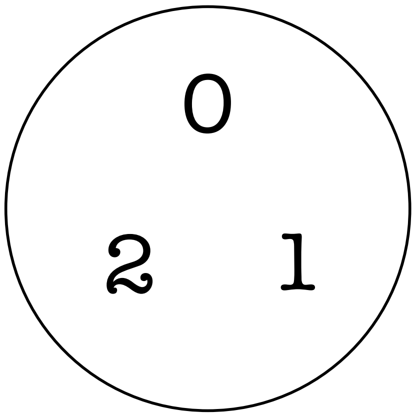


Figure 1

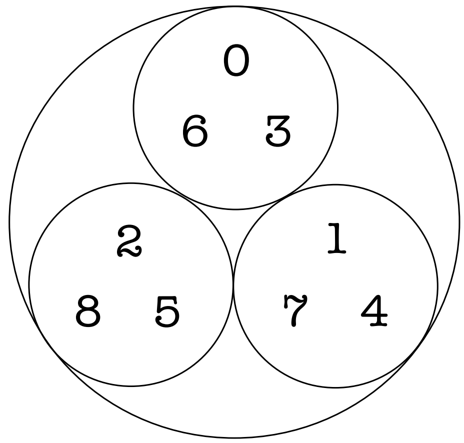


Figure 2

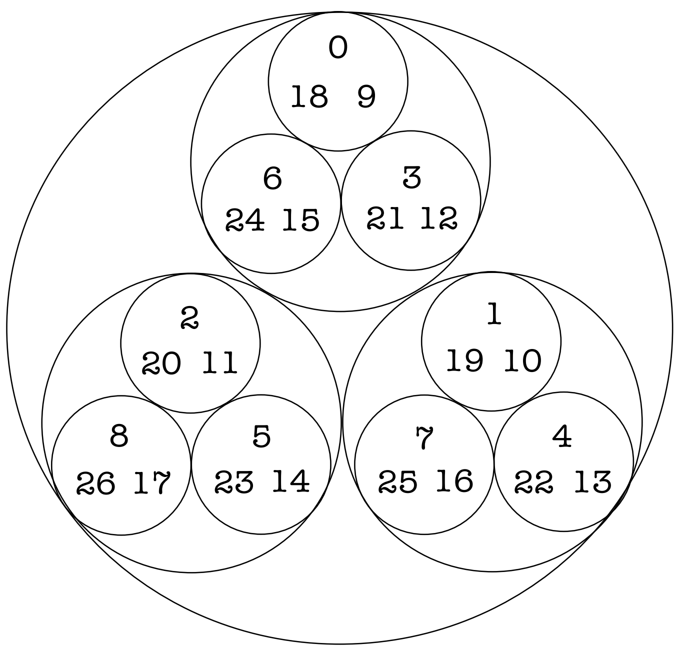
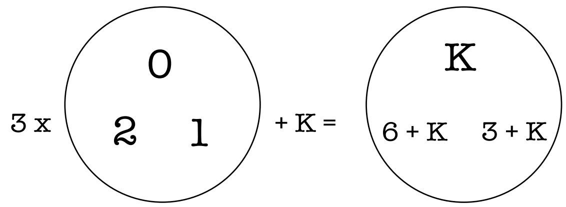


Figure 3

We construct Figure 2 from Figure 1 by replacing each number k ($k = 0, 1, 2$) in Figure 1 with $k + (3 \times \text{Figure } 1)$. In general, we construct Figure $(n + 1)$ from Figure n by replacing the number k in Figure 1 with $k + (3 \times \text{Figure } n)$:



We now define the **distance** between two nonnegative integers a, b as follows: Draw a Figure that includes both a and b, and let $N(a, b)$ be the number of circles that contain both a and b. (You might check that this number doesn't change if you use a different figure containing both a and b.) Then define the distance from a to b by

$$d(a, a) = 0, \quad \text{and} \quad d(a, b) = \frac{1}{N(a, b)} \text{ if } a \neq b.$$

For example, you can see from Figure 2 (or Figure 3) that $d(0, 3) = \frac{1}{2}$ and $d(0, 4) = 1$.

What is $d(2018, 8102)$?

Answer. $d(2018, 8102) = \frac{1}{3}$

Solution. First notice that for $a \neq b$, we have

$$N(a, b) = 1 + p,$$

where 3^p is the highest power of 3 that divides $b - a$. Now

$$8102 - 2018 = 6084 = 3^2 \cdot 676,$$

so $p = 2$. Thus, $N(2018, 8102) = 3$ and $d(2018, 8102) = \frac{1}{3}$.

To prove that $N = 1 + p$, we proceed inductively. First, notice that this is true for each pair of numbers appearing in Figure 1. Suppose it is true for each pair of numbers appearing in Figure n, and let a, b be a pair of numbers appearing in Figure $n+1$. Consider the three next-to-largest circles in Figure $n+1$. (You found the radii of these circles in Problem 15 from the written round, assuming the larger circle has radius 1 !) If a, b are not in the same one of these circles, then the difference between a and b is not a multiple of 3, and the only circle containing both of a, b is the largest one. So $N(a, b) = 1$, which agrees with the formula $N = 1 + p$ in this case, since $p = 0$. On the other hand, if a, b are in the same one of these next-to-largest circles, then we can write $a = k + 3A$ and $b = k + 3B$, where k is one of $0, 1, 2$ and A, B appear in Figure n. By construction, $N(a, b) = 1 + N(A, B)$. Since $a - b = 3(A - B)$ is divisible by one larger power of 3 than $A - B$, the claim that $N(a, b) = 1 + p$ follows by induction.

Remark. The distance defined in this question is a variation of the 3-adic distance: the 3-adic distance from a to b is $\frac{1}{3^{N(a,b)-1}}$. For a simple overview of the p-adic numbers, see Evelyn Lamb's *Scientific American* blog "Roots of Unity" at `https://blogs.scientificamerican.com/roots-of-unity/the-numbers-behind-a-fields-medalists-math/`

Problem 3 (A Farey tale)**.** For each positive integer n, the **Farey sequence** of order n, denoted $\mathfrak{F}_n$, is the (bidirectionally infinite) list of reduced fractions of denominator at most n, arranged in increasing order. For example, the terms of $\mathfrak{F}_5$ belonging to the interval $[1, 2]$ are

$$\frac{1}{1}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \frac{2}{1}.$$

Since $\sqrt{2}$ is irrational, in each $\mathfrak{F}_n$ there are consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ with

$$\frac{a}{b} < \sqrt{2} < \frac{c}{d}.$$

(For example, when $n = 5$ we have $\frac{a}{b} = \frac{7}{5}$ and $\frac{c}{d} = \frac{3}{2}$.) Find the value of $\frac{a+c}{b+d}$ when $n = 40$.

Answer. $\dfrac{58}{41}$

Solution. We list the terms of $\mathfrak{F}_n$ belonging to $[1, 2]$, for each $n = 1, 2, 3, 4, 5$:

$$\frac{1}{1}, \frac{2}{1}$$

$$\frac{1}{1}, \frac{3}{2}, \frac{2}{1}$$

$$\frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{2}{1}$$

$$\frac{1}{1}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{2}{1}$$

$$\frac{1}{1}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \frac{2}{1}.$$

After a bit of staring and some further experimentation, the following conjecture suggests itself:

If $\frac{a}{b} < \frac{c}{d}$ are consecutive fractions in some Farey sequence, then they remain consecutive in successive Farey sequences $\mathfrak{F}_n$ for all $n < b + d$. And in $\mathfrak{F}_{b+d}$, there is a unique term between $\frac{a}{b}$ and $\frac{c}{d}$, namely $\frac{a+c}{b+d}$.

We will assume the truth of the conjecture here. Proofs can be found in many books on elementary number theory; a beautifully written article treating the conjecture and closely related material is the article *Continued fractions without tears* by J. Ian Richards, an online copy of which can be found at the following address:

`http://www.maa.org/programs/maa-awards/writing-awards/continued-fractions-without-tears`

Given the truth of the conjecture, we can proceed as follows: As noted in the problem statement, $\frac{7}{5}$ and $\frac{3}{2}$ are the fractions neighboring $\sqrt{2}$ when $n = 5$. These fractions remain consecutive until $\mathfrak{F}_7$, when $\frac{10}{7}$ comes between them. Since $(\frac{10}{7})^2 = \frac{100}{49} > 2$, we see that in $\mathfrak{F}_7$, the fractions neighboring $\sqrt{2}$ are $\frac{7}{5}$ and $\frac{10}{7}$. These remain the fractions neighboring $\sqrt{2}$ until $\mathfrak{F}_{12}$, when $\frac{17}{12}$ comes between them. Since $(\frac{17}{12})^2 = \frac{289}{144} > 2$, the fractions neighboring $\sqrt{2}$ in $\mathfrak{F}_{12}$ are $\frac{7}{5}$ and $\frac{17}{12}$. These remain the fractions neighboring $\sqrt{2}$ until $\mathfrak{F}_{17}$, when $\frac{24}{17}$ comes between them. Since $(\frac{24}{17})^2 = \frac{576}{289} < 2$, the fractions neighboring $\sqrt{2}$ in $\mathfrak{F}_{17}$ are $\frac{24}{17}$ and $\frac{17}{12}$. These remain the fractions neighboring $\sqrt{2}$ until $\mathfrak{F}_{29}$, when $\frac{41}{29}$ comes between them. Since $(\frac{41}{29})^2 = \frac{1681}{841} < 2$, the fractions neighboring $\sqrt{2}$ in $\mathfrak{F}_{29}$ are $\frac{41}{29}$ and $\frac{17}{12}$. These remain the fractions neighboring $\sqrt{2}$ until $\mathfrak{F}_{41}$, and in particular these are the fractions neighboring $\sqrt{2}$ in $\mathfrak{F}_{40}$. So

$$\frac{a+c}{b+d} = \frac{41+17}{29+12} = \frac{58}{41}.$$

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