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LIMITS, AND WHAT THEY HAVE TO DO WITH CONTINUOUS FUNCTIONS

Probably the hardest thing to understand and to remember, about limits, is that the limit of a function at a point has in general **no relation whatsoever** to the value of the function at that point. [The situation is different for continuous functions of course, where the limit and the value do turn out to be equal, but not all functions are continuous, and indeed it is primarily for non-continuous functions that the concept of limit is interesting.]

Recall that it is perfectly legitimate and fair to define the values of a function in any way whatsoever: just because you have defined f(1)=6, for example, you may still define f(2) to be any thing you wish. In fact even if you have defined f(x)=6, for every real x except x=2, you still are not forced to define f(2) to be 6 also; you can define f(2)=9 if you wish. That is to say, there is no need for there to be any connection at all between the values a function has at different points of the domain. **Thus for a general function, even if you know the values it has at all but one point of the domain, you still have no way of knowing the value at that one point.**

However, there are special functions for which this is not the case. If you know a little more information about your function, it may turn out, for some special functions, that you can tell the value at one point just by knowing the values at other points. For instance if you know that the function happens to be linear, then you only need to know any two values, say f(1)=6 and f(3)=10, for example , and you can determine any other value just by using the "two-point form" of the equation for the line. Recall the way this is done: using $(x_1, y_1)=(1,6)$ and $(x_2, y_2)=(3,10)$, gives us $\Delta y= 10-6 =4$ and $\Delta x= 3-1 =2$, so the slope of our line is $m=(\Delta y/\Delta x)=(4/2)=2$, the equation is y=2x+b, and plugging in (x,y)=(1,6) for example gives 6=2+b, so that b=4. Thus the function has the formula f(x)= 2 x + 4, and we can determine the value f(2), (or f(x) for any other x we choose), just by plugging into this formula. e.g. f(2)=2(2)+4=8, in this case.

There are other functions also such that the values at one point can be determined from knowing the values at other points. Take a "quadratic" function, i.e. one whose graph is known to be a parabola. This time to figure out what f(2) is we need to know the values of f(x) for three other points: say we know for example that f(0)=1, f(1)=0, and f(3)=10. Then since the function is quadratic we know it has an equation like $f(x)=ax^2+bx+c$. Plugging in the three values above gives the three equations 1=c, 0=a+b+c,

$10=9a+3b+c$. Substituting $c=1$ into the other two gives $0=a+b+1$, $10=9a+3b+1$. From the first of these equations we can deduce that $a=(-b-1)$, so that the second equation becomes $10=9(-b-1)+3b+1= (-6b-8)$, so that $6b=(-18)$, and $b= -3$. Thus $a=(3-1)=2$, and $f(x)=2x^2-3x+1$. Hence $f(2)=2(2)^2-3(2)+1=8-6+1=3$.

Now it is easy to see that the higher the degree of the polynomial the more points at which you are going to need to know the values, before you can figure out the value at any other point, (because higher degree polynomials have more coefficients in their formulas that you need to solve for).

There are also functions whose formulas even have "infinite" degree, such as the infinite geometric series $f(x)=1+x+x^2+x^3+x^4+..........$, which continues "forever", and such functions were studied very closely by the old masters of the calculus. If you knew only that your function had an infinite formula like $f(x)= a+bx+cx^2+dx^3+ex^4+..........$, then you would need to know presumably an infinite number of other values before you could solve for an unknown value like $f(2)$. On the other hand, if you are dealing with a function like the one defined by the rule : $f(x)=5$, if $x\neq 0$, and $f(0)$ = the number of stars in the sky at 8 pm tonight, then just because you know the values of f at every non-zero number does not help you find out the value at $x = 0$.

Now Newton was concerned with a very special type of function, the slope function for secant lines to a graph, passing through a given point. If f is a given function, let us define the function m as follows: when $x\neq a$, define $m(x)$=the slope of the secant line joining $(a,f(a))$ to the point $(x,f(x))$, and define $m(a)$= the slope of the tangent line to the graph of f at $(a,f(a))$. Now we are in a situation somewhat like those above. We know that when $x\neq a$ the slope m of the secant line joining $(a,f(a))$ to $(x,f(x))$ is given by the formula $(\Delta y/\Delta x)=(\Delta f/\Delta x)=\{[f(x)-f(a)]/(x-a)\}$. But this formula does not give the slope of the tangent line when $x=a$. So we know what $m(x)$ is for all $x\neq a$, but we do not know the value of $m(a)$.

Now just as in the cases above where we were able to figure out the value of $f(2)$ from knowing a lot of other values of f, Newton probably thought he should be able to figure out what $m(a)$ was from knowing all those other values of $m(x)$ for $x \neq a$. <u>He had to try to figure out a way to describe what the value $m(a)$ of the function m should be at a, entirely in terms of the values $m(x)$ for values of x different from a.</u> Now by looking at the graph of a reasonable curve Newton must have noticed that as x came closer to a, the secant line through $(a,f(a))$ and$(x,f(x))$ came closer and closer to the tangent line at $(a,f(a)$. (In fact he probably got this idea from Euclid, Book III, Prop. 16, where it is proved that the tangent to a circle is approximated arbitrarily well by secant lines). Thus if he could just figure out, from looking at the values of m(x) with $x \neq a$, what one value they were getting closer to as x approached a, he would have figured out what m(a) was. But how to do it?

What he came up with was what we call the "limit of the values f(x) as x approaches a". Newton understood exactly what he meant by this but had trouble explaining it to other people. He said something like this: if there is a number L, such that the values of f(x) can be made as close as desired to L simply by taking x close enough to a, then L is said to be the limit of the values f(x) as $x \to a$. Note that there is at most one such number L, since the values of f(x) cannot be simultaneously arbitrarily close to two different numbers.

Here is the actual precise definition as given later by the nineteenth century mathematicians: $\lim_{x\to a} f(x)=L$ if and only if: to every positive number e, there corresponds a positive number ∂, such that every x with $0<|x-a|<\partial$, satisfies $|f(x)-L|<e$.

This definition is a little complicated, but you can see at least that the value f(a) is not mentioned in the definition. i.e.: **the limit of f(x) as x approaches a, is a number which is entirely determined by the values f has at points <u>different from</u> a.**

Thus, in determining what the limit of f(x) is, as x approaches a, you don't even consider the value f(a). If you did, the method would of course be of no use in helping you to figure out what that value is, which is the whole point. So how do you find out what that value is? You can use the same method we used above to handle linear and quadratic functions, except you use instead <u>continuous</u> functions. A continuous function is a function for which the value at a and the limit at a are the same. Thus for a continuous function the value f(a) is determined by knowing the values of all f(x) with $x \neq a$. Our remarks above about the slope function m(x) associated to a "reasonable" function f amount to saying that m is continuous at a.

Here's how to take advantage of that: recall again that in the linear example above we were studying a function f(x) about which we knew a couple of things, (i): the function f was linear; and (ii): f(1)=6 and f(3)=10. Then we worked a little and came up with a function g(x)=2x+4. This function also has g(1)=6, and g(3)=10, as you can check by plugging in and evaluating. Thus we had two functions g and f and we knew both of them were linear and both of them had the same values at x=1 and at x=3. Then we said that since a line is determined by any two points on it that f and g must have the same values everywhere. Thus to compute f(2), we just computed $g(2) = 2(2)+4 = 8$.

We did the same thing in the quadratic example: that is we had a function f we didn't know much about except that: (i) f was quadratic, and (ii) $f(0)=1$, $f(1)=0$, and $f(3)=10$. Then we worked a bit and cooked up an explicit function $g(x)=2x^2-3x+1$, for which also $g(0)=1$, $g(1)=0$, and $g(3)=10$. Then we had two functions f and g, and we knew that both were quadratic, and both had the same values at x=0, x=1, and x=3. Since we knew that a parabola is determined by any three points on it, we deduce that the two functions f and g must agree everywhere, and so to compute f(2), we could use g, which gives $f(2)=g(2)=2(2)^2-3(2)+1 = 3$.

We can handle the secant example in the same way using the notion of continuity. I.e., just as a <u>quadratic</u> function's value at a is determined by knowing any three other values, <u>the value of a continuous function f at a is determined by knowing the values of f(x) at all other values near a</u>. That is , if a lies in the interval (b,c) and if f is continuous at a, and if we know the values of f(x) at all other points of the interval except at a, then the value at a is determined too. This gives us the following principle:

Theorem: Let a be point lying in the interval (b,c), and let f and g be two functions both defined on that interval. If f and g are both continuous at a, and if $f(x) = g(x)$ for every $x \neq a$, then $f(a) = g(a)$ also.

This principle can be used to compute slopes as follows: let f be a function whose graph is a smooth curve and for which we want to compute the slope of the tangent line to the graph at (a,f(a)). As above, consider the function m(x) whose value at any x with $x \neq a$ is the slope of the secant line joining (a,f(a)) to (x,f(x)), and whose value at x=a is the slope of the tangent line to the graph of f at (a,f(a)). We claim m is a continuous function. All we will say to justify this is that if you look at the graph of any nice curve we see that the secant line through (a,f(a)) and (x,f(x)) approaches the tangent line at (a,f(a)) as x approaches a. So let's accept that indeed m is a continuous function.

Moreover we have a formula for the values of m that works at least for $x \neq a$, namely $\{[f(x)-f(a)]/(x-a)\}=m(x)$, for $x \neq a$. Thus if we can cook up some function g which is continuous at a, and which has the same values as this formula for $x \neq a$, then that g must agree with our slope function m everywhere. Thus to compute m(a), it would suffice to compute g(a). <u>Note we are not applying the theorem above to the functions f and g, but to the functions m and g instead.</u>

Here is a typical example. Let $f(x) = x^2$. Then m(x) is given by the formula $\Delta f/\Delta x = (x^2-a^2)/(x-a,)$ at least for $x \neq a$. But m is also continuous at x = a, whereas this formula is not, so this formula is no good for computing the value m(a). However if we factor the formula, we get $\Delta f/\Delta x = [(x+a)(x-a)]/(x-a)$. If we now cancel the factors of (x-a), we come up with the function g(x) = (x+a). This g is a continuous function everywhere including at a, since it is a polynomial. Moreover g(x) = m(x), for all $x \neq a$, since when $x \neq a$ they both equal the formula $\Delta f/\Delta x$. Thus by our theorem, since both m and g are continuous at a, and are equal everywhere near a, they must also be equal at a. Hence to compute m(a) we use g(a)=(a+a) = 2a. That gives the slope of the graph of x^2 at (a,f(a)) as 2a.

Another example is to compute the slope of the tangent line to the graph of $f(x) = (x)^{1/2}$. Here $\Delta f/\Delta x = [x^{1/2}-a^{1/2}]/(x-a)$. This gives the value of m(x) for $x \neq a$, but not at x = a. If we rationalize the expression we get $(x-a)/\{(x-a)(x^{1/2}+a^{1/2})\}$, so when we cancel we get g(x) = $1/(x^{1/2}+a^{1/2})$. This is continuous at x = a, and equals m(x) for all $x \neq a$, hence must also equal m for x = a. Thus we can compute m(a) = g(a) = $1/(a^{1/2}+a^{1/2}) = 1/(2a^{1/2}) = (1/2)a^{-1/2}$, at least for $a \neq 0$.

We can use continuous functions also to compute limits of functions which themselves are not continuous. I.e. for a function f that is not necessarily continuous at a, the "limit of f(x) as x approaches a", is the value at x=a, of any continuous function g which is equal to f away from a. In other words:

<u>**Theorem: If f is a function defined on some interval containing a, and if g is a continuous function defined on that same interval , and if f(x)=g(x) for all x in the interval with x≠a, then f does have a limit as x approaches a, and in fact that limit is g(a).**</u>

i.e. $\{\lim_{x\to a} f(x)\}=g(a)$, provided g agrees with f for x≠a and g is continuous at x=a.

Note that the theorem is still true no matter what the value f(a) is, and is true even if f is undefined at a; i.e. <u>the value f(a) plays no role at all in determining the limit of f(x) as x approaches a.</u>

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Thus to compute the limit of f(x) as x approaches a, in case f is a function whose value at a we may not even know, the only way we have now is to find some continuous function g which agrees with f for all x with $x \neq a$, and then to compute g(a). But what do we do when we can't find such a function? How do we compute the limit then?

One way is to use the **"squeeze principle".** The idea is that if two continuous functions m and g are defined on the same interval (b,c) containing a, and if we know that $m(x) \leq g(x)$ for all $x \neq a$. then it follows that also $m(a) \leq g(a)$. Thus suppose we have a function f whose limit we want to compute at x=a, and we cannot find a continuous function m that equals f away from a, but we do the best we can. Say we can find <u>two</u> continuous functions g and h such that we have $g(x) \leq f(x) \leq h(x)$ for all x with $x \neq a$, and such that $g(a)=h(a)$. Then whatever that elusive continuous function m was which equaled f for $x \neq a$, it would have to lie between g and h. i.e. we would have to have $g(x) \leq m(x) \leq h(x)$ for all $x \neq a$, and hence also $g(a) \leq m(a) \leq h(a)$. Then since $g(a)=h(a)$, we would have $g(a)=m(a)=h(a)$. Thus we don't need to find m, just g and h.

To be precise:

Squeeze Principle: If g,f,h are three functions defined on the same interval containing a, (except f need not be defined <u>at</u> a), and if:
(i) $g(x) \leq f(x) \leq h(x)$ for all x with $x \neq a$,
(ii) g and h are continuous at x=a,
(iii) $g(a)=h(a)$,
then the limit of f as $x \to a$ exists, and $g(a) = \lim_{x \to a} f(x) = h(a)$.

Notice again that nothing is said about f(a), and in fact nothing is known about f(a) from these assumptions.

<u>We apply this next to differentiate sin(x)</u>. (Remember, this is the derivative we could not do by Descartes' method.) Consider the difference quotient for the function sin(x) at a=0, i.e. $(\Delta f/\Delta x)=[\sin(x)-\sin(0)]/(x-0)$, which simplifies to $\sin(x)/x$. We want to compute this limit as $x \to 0$. Since we don't see how to simplify $\sin(x)/x$ to a function which is equal to it for $x \neq 0$, but which is also continuous at x=0, we look for the two "squeeze" functions instead.

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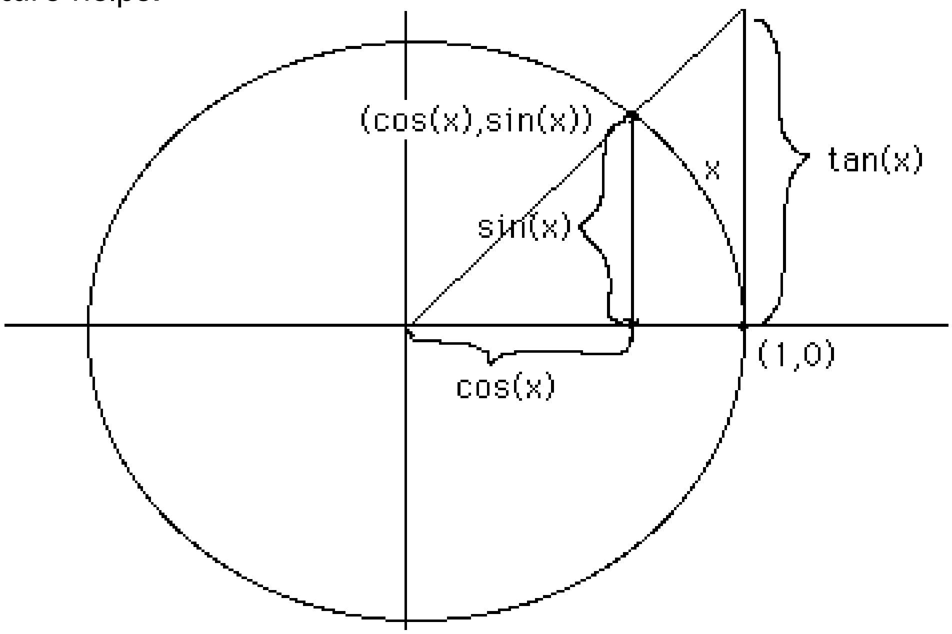
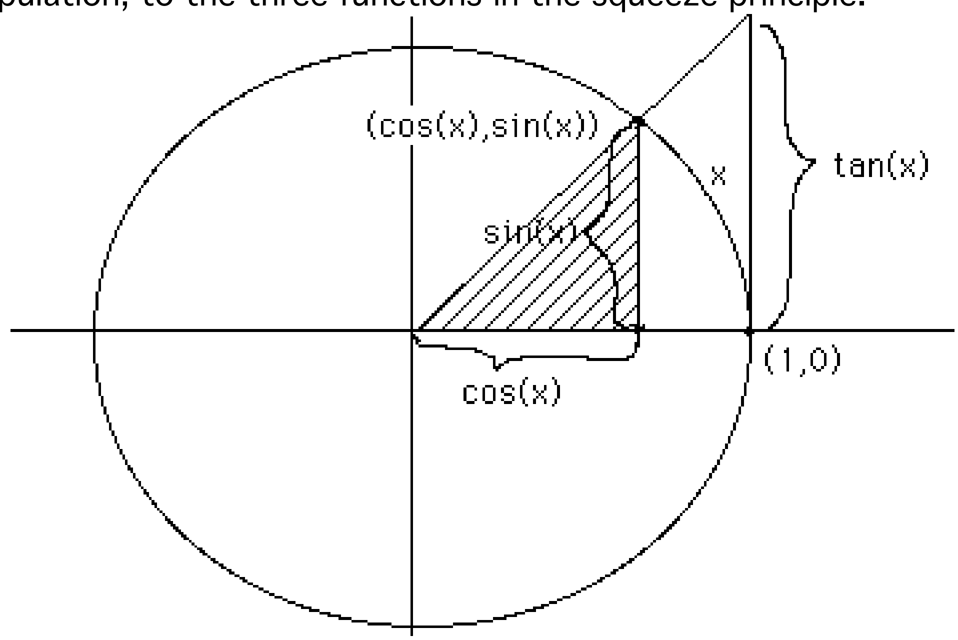
A picture helps:



Here x is the angle in radians, i.e. x is the length of the arc of the unit circle between the point (1,0) and the point (cos(x),sin(x)), as usual in the "circular function" approach to defining sine and cosine. We want to examine the three areas of this picture which are shaded below. Notice that each area is larger than the one before. The three area functions are going to give rise, after some manipulation, to the three functions in the squeeze principle.



The shaded triangular area above is (1/2)(base)(height)=(1/2)sin(x)cos(x).

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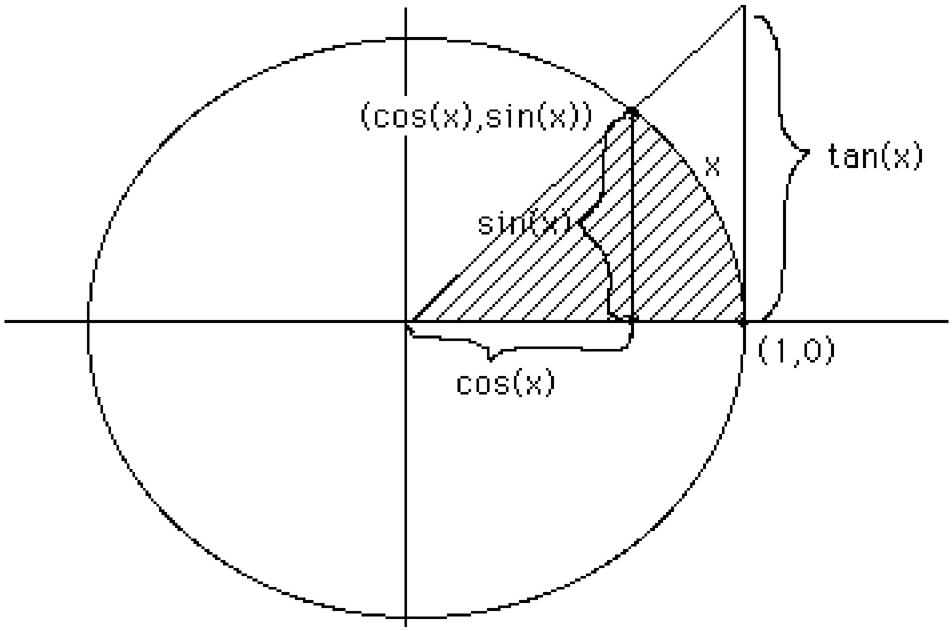
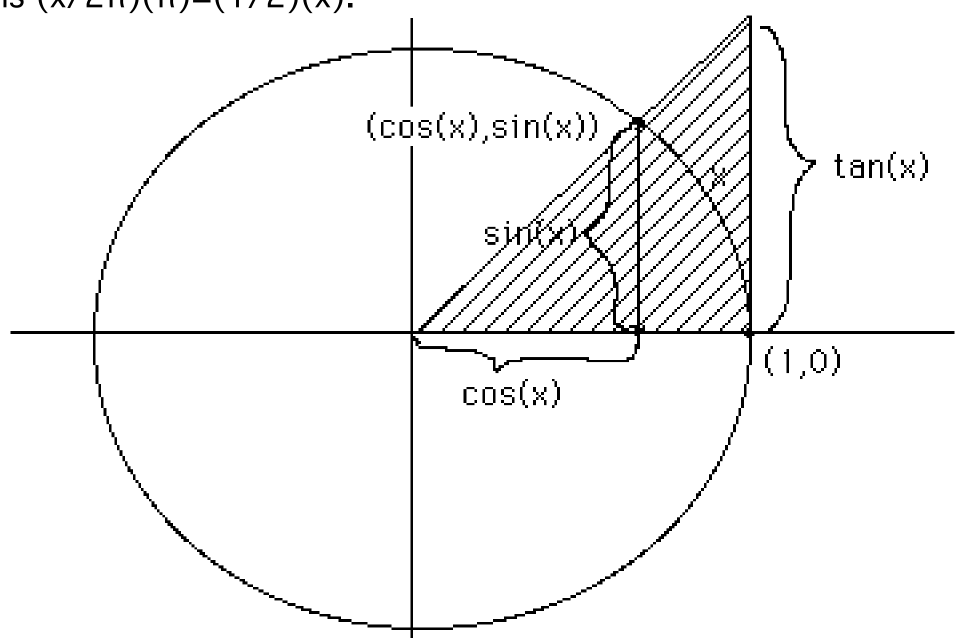


Here the shaded area is that of an arc of the unit circle spanning x radians. Since the whole unit circle spans 2π radians and has area $\pi r^2=\pi$, the shaded area is $(x/2\pi)(\pi)=(1/2)(x)$.



Here the triangle has (base)=1, (height)=tan(x), and thus (area)=(1/2)tan(x). Thus we get the inequalities:

$$(1/2)\sin(x)\cos(x) \le (1/2)x \le (1/2)\tan(x).$$

Multiplying by 2 and taking reciprocals reverses the inequalities, and gives:

$$(1/\tan(x)) \leq (1/x) \leq \{1/[\sin(x)\cos(x)]\},$$

which is equivalent to:

$$[\cos(x)/\sin(x)] \leq (1/x) \leq \{1/[\sin(x)\cos(x)]\}.$$

Now multiply though by sin(x), and we get:

$$\cos(x) \leq \{\sin(x)/x\} \leq \{1/\cos(x)\}.$$

This gives us the squeeze play we want. Thus let g(x)=cos(x), let h(x)={1/cos)x)}, and let f(x)=sin(x)/x. Then the hypotheses of the squeeze principle are all satisfied since: cos is a continuous function, and 1/cos is also continuous at x=0, (since $\cos(0)=1\neq0$), and since last of all cos(0)=1/cos(0)=1. Thus we can apply the principle and conclude that $\lim_{x\to0}[\sin(x)/x]=1$, also. Thus we have computed the derivative of sin(x) at x=0, namely sin'(0)=1.

Now to compute the derivatives of sin(x) at other points we use our old friends the addition formulas from trigonometry. Recall in particular that sin(x+y) = sin(x)cos(y)+cos(x)sin(y). Now we apply that. We have by definition that $\sin'(x) = \lim_{h\to0}[\sin(x+h)-\sin(x)]/h$. By the addition formula this becomes the limit of [sin(x)cos(h)-sin(x)]/h + [sin(h)cos(x)]/h. As h approaches 0, the second term approaches cos(x) times the limit of sin(h)/h, so by what we just proved the second term has limit equal to cos(x).

The first term, on the other hand, approaches sin(x) times the limit of [cos(h)-1]/h, as $h\to0$. What is this limit? Just "rationalize it" by multiplying by [cos(h)+1]/[cos(h)+1] to get $[\cos^2(h)-1]/h(\cos(h)+1)$, and then notice that $\cos^2(h)-1= -\sin^2(h)$, (by using everyone's favorite identity $\cos^2+\sin^2=1$), so the limit becomes $[-\sin^2(h)]/h(\cos(h)+1)=[\sin(h)/h][-\sin(h)/ (\cos(h)+1)]$.

Since the first factor on the right approaches 1 and the second factor approaches 0/2=0, the product approaches 0. Finally, adding the two limits we have found gives cos(x)+0=cos(x). Hence we have computed the derivative sin'(x)=cos(x). Hooray !!

Exercise: Prove that cos'(x)= -sin(x).