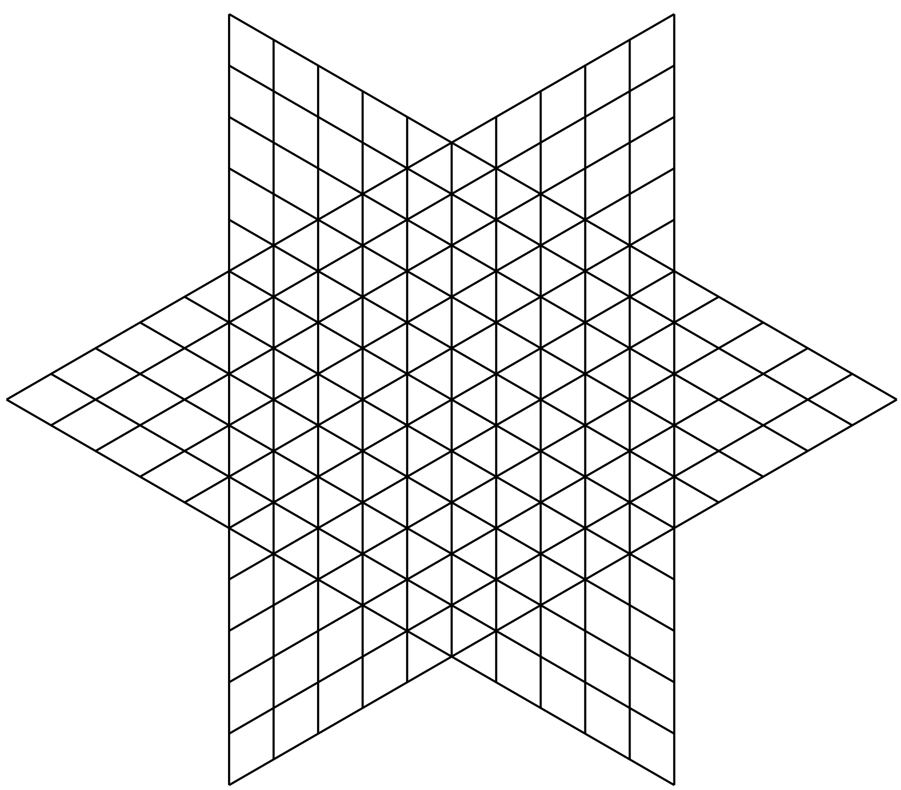
Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
October 20, 2018

**No calculators are allowed on this test.** You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

**Problem 1** (Triangles and tribulations)**.** How many triangles are in the figure below? Only count those triangles whose edges lie on the lines shown.



**Problem 2** (Circular reasoning)**.** Figures 1–3 below are the first three in a sequence of figures.

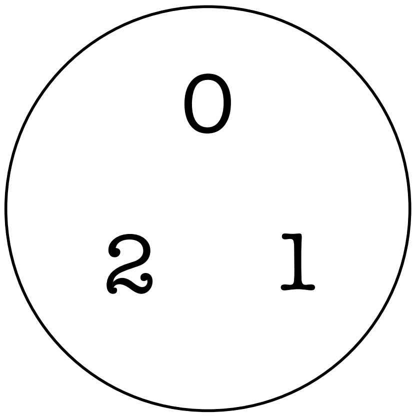



Figure 1

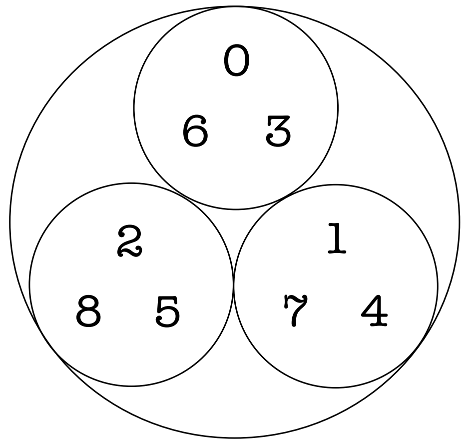



Figure 2

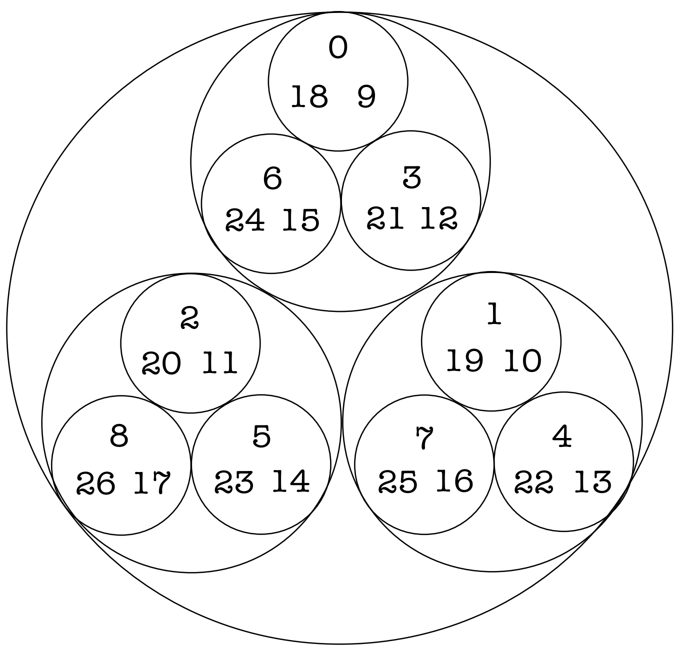
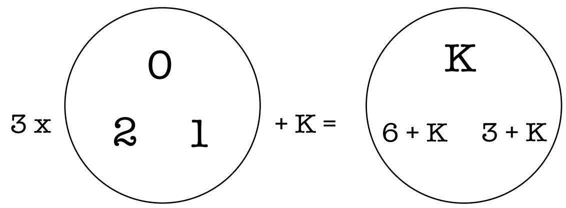



Figure 3

We construct Figure 2 from Figure 1 by replacing each number $k$ ($k = 0, 1, 2$) in Figure 1 with $k + (3 \times \text{Figure } 1)$. In general, we construct Figure $(n + 1)$ from Figure $n$ by replacing the number $k$ in Figure 1 with $k + (3 \times \text{Figure } n)$:




We now define the **distance** between two nonnegative integers $a, b$ as follows: Draw a Figure that includes both $a$ and $b$, and let $N(a, b)$ be the number of circles that contain both $a$ and $b$. (You might check that this number doesn't change if you use a different figure containing both $a$ and $b$.) Then define the distance from $a$ to $b$ by

$$d(a, a) = 0, \quad \text{and} \quad d(a, b) = \frac{1}{N(a, b)} \text{ if } a \neq b.$$

For example, you can see from Figure 2 (or Figure 3) that $d(0, 3) = \frac{1}{2}$ and $d(0, 4) = 1$.

What is $d(2018, 8102)$ ?

**Problem 3** (A Farey tale)**.** For each positive integer $n$, the **Farey sequence** of order $n$, denoted $\mathfrak{F}_n$, is the (bidirectionally infinite) list of reduced fractions of denominator at most $n$, arranged in increasing order. For example, the terms of $\mathfrak{F}_5$ belonging to the interval $[1, 2]$ are

$$\frac{1}{1}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \frac{2}{1}.$$

Since $\sqrt{2}$ is irrational, in each $\mathfrak{F}_n$ there are consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ with

$$\frac{a}{b} < \sqrt{2} < \frac{c}{d}.$$

(For example, when $n = 5$ we have $\frac{a}{b} = \frac{7}{5}$ and $\frac{c}{d} = \frac{3}{2}$.) Find the value of $\frac{a+c}{b+d}$ when $n = 40$.

RETURN THIS SHEET

**Team ID:**

Team name:

Answer 1:

Answer 2:

Answer 3: