Real Analysis Qualifying Examination
January 2013

There are five problems, each worth 20 points. Give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Suppose $\{a_n\}$ is a sequence of real numbers and define $c_n = \dfrac{a_1 + \cdots + a_n}{n}$.
   (a) Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} c_n = L$ also.
   (b) Is the converse to part (a) true? Give either a proof or counterexample.

2. Let $O \subseteq \mathbb{R}^n$ be open with $O^c = \mathbb{R}^n \setminus O \neq \emptyset$. If $E \subseteq O$, not necessarily Lebesgue measurable, and $E_k = \{x \in E : \ d(x, O^c) \geq 1/k\}$, prove that $\lim_{k\to\infty} m_*(E_k) = m_*(E)$, where m_* denotes the exterior Lebesgue measure on $\mathbb{R}^n$.

3. Let $0 < p < \infty$ and $f : \mathbb{R}^n \to \mathbb{C}$ be a Lebesgue measurable function. For each $t > 0$, let
$$\lambda_f(t) = m(\{x : |f(x)| > t\})$$
where m denotes Lebesgue measure on $\mathbb{R}^n$. Recall (no proofs required!) that
$$\sup_{t>0} t^p \lambda_f(t) \leq \int_{\mathbb{R}^n} |f(x)|^p \, dx = p \int_0^\infty t^{p-1} \lambda_f(t) \, dt.$$
   (a) Show that if $(1+t)^4 \lambda_f(t) \leq C$ for all $t > 0$, then $f \in L^p(\mathbb{R}^n)$ for all $0 < p < 4$.
   (b) Give an example of a measurable function $f : \mathbb{R} \to \mathbb{C}$ that is <u>not</u> in $L^4(\mathbb{R})$, but for which
$$\sup_{t>0} t^4 \lambda_f(t) \leq C.$$

4. Let $1 < p < \infty$. Let f_k be a sequence of functions in $L^p(\mathbb{R}^n)$ such that for each $k = 1, 2, \ldots$
   (i) $\|f_k\|_p = 1$
   (ii) for almost every $x \in \mathbb{R}^n$ we have either $f_k(x) = 0$ or $f_k(x) \geq k$.

   Prove that
$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x) g(x) \, dx = 0$$
   for any $g \in L^q(\mathbb{R}^n)$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

5. (a) Let $\{e_k\}_{k=1}^\infty$ be an orthonormal sequence in a Hilbert space H. Prove that
$$\lim_{k\to\infty} \langle h, e_k \rangle = 0$$
   for all $h \in H$.
   (b) Let $\{e_k\}_{k=1}^\infty$ be a complete orthonormal basis for the specific Hilbert space $L^2([0,1])$ equipped with its usual inner product $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)} \, dx$.

   Prove that if $f \in L^2([0,1])$, $\|e_k\|_{L^\infty([0,1])} \leq C$ uniformly in k, and $\sum_{k=1}^\infty |\langle f, e_k \rangle| < \infty$, then
$$S_N f(x) = \sum_{k=1}^N \langle f, e_k \rangle e_k(x)$$
   converges *pointwise* to $f(x)$ as $N \to \infty$ for almost every x in $[0,1]$.