NAIVE INTRODUCTION TO ALGEBRAIC GEOMETRY: THE GEOMETRY OF RINGS

I used to say algebraic geometry is the study of the "geometry of polynomials". Now I sometimes call it the "geometry of rings". I also feel that algebraic geometry is defined more by the objects it studies than the tools it uses. The naivete in the title is my own.

I. BASIC TOOL: RATIONAL PARAMETRIZATION

Algebraic geometry is a generalization of analytic geometry - the familiar study of lines, planes, circles, parabolas, ellipses, hyperbolas, and their 3 dimensional versions: spheres, cones, hyperboloids, ellipsoids, and hyperbolic surfaces. The essential common property these all have is that they are defined by polynomials. This is the defining characteristic of classical algebraic sets, or varieties - they are loci of polynomial equations.

A further inessential condition in the examples above is that the defining polynomials have degree at most 2 and involve at most 3 variables. This limitation arose historically for psychological and technical reasons. Before the advent of coordinates, higher dimensions could not be envisioned or manipulated, and even afterwards it was commonly felt that space of more than 3 dimensions did not "exist" hence was irrelevant.

The dimension barrier was lifted by Riemann and Italian geometers in the 19th century such as C. Segre, who realized that higher dimensions could be useful for the study of curves and surfaces. Riemann's use of complex coordinates for plane curves simplified their study, and embedded a curve of genus g in an abelian variety (complex torus) of dimension g. Segre exploited the fact that some surfaces in 3 space (quartics with a double conic) were projections of simpler ones embedded in 4 space (intersections of two quadrics).

One historical reason for restricting attention to equations in (X,Y) of degree at most 2 is a limitation of the basic method of "parametrization", expressing a locus by an auxiliary parameter. E.g. the curve $X^2 + Y^2 = 1$ can be parametrized by the variable t by setting $X(t) = 2t/[1+t^2]$, $Y = [1-t^2]/[1+t^2]$. This substitutiion, along with $dX = 2[1-t^2]dt/[1+t^2]^2$, allows one to simplify the integral of $dX/\mathrm{sqrt}(1-X^2)$, to that of $2dt/[1+t^2] = 2d[\arctan(t)]$.

The cubic $Y^2=X^3$ can also be parametrized, say by $X = t^2$, $Y = t^3$. But to simplify in this way the integral of $dX/\mathrm{sqrt}(1-X^3)$, requires us to parametrize the cubic $Y^2 = 1-X^3$, a problem which is actually impossible. These questions were considered first by the Bernoullis, and resolved by new ideas of Abel, Galois, and especially Riemann as follows. (Interestingly, in three variables the difficulty arises in degree 4, and 19th century geometers already knew how to parametrize most cubic surfaces.)

II. NEW METHODS FOR PLANE CURVES: TOPOLOGY and COMPLEX ANALYSIS

Riemann associated to a plane curve $f(X,Y)=0$ its set of complex solutions, compactified and desingularized. This is its "Riemann surface", a real topological 2 manifold with a complex structure obtained by a branched projection onto the complex line. For instance the curve $y^2 = 1-X^3$ becomes its own Riemann surface after adding one point at infinity, making it a topological torus. Projection on the X coordinate is a 2:1 cover of the extended X line, branched over infinity and the solutions of $1-X^3 = 0$.

This association is a functor, i.e. a non constant rational map of plane curves yields an associated holomorphic map of their Riemann surfaces, in particular a topological branched cover. Riemann assigns to a real 2 manifold its "genus" (the number of handles), and calculates that branched covers cannot raise genus, and the only surface of genus zero is the sphere = the Riemann surface of the complex t line. Hence if the Riemann surface of a plane curve has positive genus, it cannot be the branched image of the sphere, hence the curve cannot be parametrized by the coordinate t.

Riemann also proved a smooth plane curve of degree d has genus $g = (d-1)(d-2)/2$, so smooth cubics and higher degree curves all have positive genus and hence cannot be parametrized. He proved conversely that any curve whose Riemann surface has genus zero can be parametrized, e.g. hyperbolas, circles, lines, parabolas, ellipses, or any curve of degree < 3. Moreover a singularity, i.e. a point where the curve has no tangent line, like (0,0) on Y^2 = X^3, lowers the genus during the desingularization process, and this is why such a "singular" cubic can be parametrized.

One also obtains a criterion for any two irreducible plane curves to be rationally isomorphic, namely their Riemann surfaces should be not just topologically, but holomorphically isomorphic. By representing a smooth plane cubic as a quotient of the complex line C by a lattice, using the Weierstrass P function, one can prove that many complex tori are not holomorphically equivalent, by studying the induced map of lattices. It follows that there is a one parameter family of smooth plane cubics which are rationally distinct from each other.

This shows briefly the power and flexibility of topological and holomorphic methods, which Riemann largely invented for this purpose, an amazing illustration of thinking outside traditional confines.

III. RINGS and IDEALS

To go further in the direction of arithmetic questions, one would like more algebraic techniques, applicable to fields of characteristic p, algebraic number fields, rings of integers, power series rings,.... One can pose the question of isomorphism of plane curves algebraically, using ring theory, as follows. Since all roots of multiples of the polynomial f vanish on the zero locus of f, it is natural to associate to the curve V:{f=0} in k^2, the ideal rad(f) = {g in k[X,Y]: some power of g is in (f)}. Then the quotient ring R = k[X,Y]/rad(f) is the ring of polynomial functions on V. Moreover if p is a point of V, evaluation at p is a k algebra homomorphism R-->k with kernel a maximal ideal of R. In case k is an algebraically closed field, like C or the algebraic numbers, this is a bijection between points of V and maximal ideals of R.

In fact everything about the plane curve V is mirrored in the ring R in this case, and two irreducible polynomials f,g, in k[X,Y], define isomorphic plane curves if and only if their associated rings R and S are isomorphic k algebras. Indeed the assignment of R to V is a "fully faithful functor", with algebraic morphisms of curves corresponding precisely to k algebra maps of their rings. To recover the points from the ring one takes the maximal ideals, and to recover a map on these points from a k algebra map, one pulls back maximal ideals. (Since these rings are finitely generated k algebras and k is algebraically closed, a maximal ideal pulls back to a maximal ideal.) Any pair of generators of the k algebra R defines an embedding of V in the plane.

Similarly, if f (irreducible) in k[X,Y,Z] defines a surface V:{f=0} in k^3, (k still an algebraically closed field), then not only do points of V correspond to maximal ideals of R = k[X,Y,Z]/(f), but irreducible algebraic curves lying on V correspond to non zero non maximal prime ideals in R. Again this is a fully faithful functor, with polynomial maps corresponding to k algebra maps. In particular the pullback of maximal ideals is maximal, but now the pullback of some non maximal ideals can also be maximal, i.e. some curves can collapse to points under a polynomial map.

To give the algebraic notion full flexibility, in particular to embrace non Jacobson rings with too few maximal ideals to carry all the desired structure, Grothendieck understood one should discard the restriction to rings without radical and expand the concept of a "point", to include irreducible subvarieties, i.e. consider all prime ideals as points, as follows.

IV. AFFINE SCHEMES

If R is any commutative ring with 1, let V (= "specR") be the set of all prime ideals of R, with a topological closure operator where the closure of a set of prime ideals is the set of all prime ideals containing the intersection of the given set of primes. (Intuitively, each prime ideal contains the functions vanishing at the corresponding point, so their intersection is all functions vanishing at all the points of the set, and the prime ideals containing this intersection hence are all points on which that same set of functions vanishes. So the closure of a set is the smallest algebraically defined locus containing the set.) This closure operator defines the "Zariski topology" on V.

Now any ring map defines a morphism of their spectra by puling back prime ideals, and in particular a morphism is continuous, although this alone says little since the Zariski topology is so coarse. Notice now maximal ideals may pull back to non maximal ones, e.g. under the inclusion map Z-->Q of integers to the rationals, taking the unique point of specQ to a dense point of specZ. Maximal ideals now correspond to closed points, and in particular there are usually plenty of non closed points. Intuitively, every irreducible subvariety has a dense point, and together these "points", one for each irreducible subvariety, give all the points of specV.

If K is a ring, a "K valued point" of V is given by a ring homomorphism R-->K, not necessarily surjective. E.g. if K is a field, the pullback of the unique maximal ideal of K is a not necessarily maximal, prime ideal P of R, the K valued point. Even if the point is closed, i.e. if P is maximal, we get information on which maximal ideals correspond to points with coefficients in different fields. If say k = the real field, and f is a polynomial over k, then a k algebra map g:k[X,Y]/(f)-->k has as kernel a maximal ideal corresponding to a point of {f=0} in k^2, i.e. a point of {f=0} in the usual sense, with real coefficients.

The coordinates of this point are given by the pair of images (g(X),g(Y)) in k^2 of the variables X,Y, under the algebra map g, which after all is evaluation of functions at our point. But if say f = Y-X^2, the map from k[X,Y]/(f) -->C taking X to i, and Y to -1, corresponds to the C (complex) - valued point (i,-1), in C^2 rather than k^2.

More generally, if I is any ideal in Z[X1,...,Xn] generated by integral polynomials f1,...fr, and A is a ring, a ring homomorphism Z[X1,...,Xn]/I -->A takes the variables Xj to elements aj of A such that all the polynomials fi vanish at the point of A^n with cordinates (a1,...an). I.e. the map defines an "A valued point " of the locus defined by I. E.g. if M is a maximal ideal of R,we can always view the coordinates of the corresponding point in the residue field R/M, i.e. the point M of specR is "R/M valued".

This approach lets us recover tangent vectors too, in case say of a variety V with ring R = k [X1,...,Xn]/I, where radI = I, and k is an algebraically closed field. Consider the ring S = k[T]/(T^2), with unique maximal ideal (t) generated by the nilpotent element t. Then we claim tangent vectors to V correspond to S valued points (over spec(k)), i.e. to k algebra maps R-->S. E.g. if R = k[X], and we map R-->S by sending X to a+bt, then the inverse image of the maximal ideal (t) is the maximal ideal (X-a), and two elements of (X-a) have the same image in S if and only if they have the same derivative at X=a. I.e. this maps represent the point a of the X line, and the tangent vector w at a, such that the directional derivative of X-a along w equals b. Thus S valued points of V are points of the "tangent bundle" of V.

V. SCHEMES

One next defines a scheme as a space with an open cover by affine schemes, by analogy with topological manifolds, which have an open cover by affine spaces. For this we need to be able to glue affine schemes along open subsets, so we need to understand the induced structure on an open subset of V = specR. A Zariski closed set consists of primes containing a given collection of elements {fj} of R, which is the intersection for all fj of the set of primes containing fj.

Since the complementary open set is the union of the open sets V(fj) of primes not containing fj, a basis for the Zariski topology of specR is given by the open sets of form V(f) = {primes P in specR with f not in P}. Intuitively this is the set of points where the element f does not vanish. (The analogy is with a "completely regular" topological space whose closed sets are cut out by continuous real valued functions.)

On the set V(f), where V = specR, the most natural ring is R(f) = {g/f^n: g in R, n a non negative integer}/{identification of two fractions if their cross product is annihilated by a non neg. power of f}. I.e. since powers of f are now units, anything annihilated by a unit must become zero, so g/f^n = h/f^m if for some s, $f^s[gf^m - hf^n] = 0$ in R. Intuitively these are rational functions on V which are regular in V(f). Since prime ideals of R(f) correspond to primes of R not containing f, the open set V(f) = specR(f), is also an affine scheme.

This construction defines an assignment of a ring to each basic open set V(f) in V, and a ring map for each inclusion of open sets, such that the trivial inclusion yields the identity ring map, and compositions of inclusions yield compositions of ring maps. I.e. it defines a (pre)sheaf of rings on a basis for V, and hence a sheaf of rings on all of V, by a standard extension construction. This sheaf is called O, perhaps in honor of the great Japanese mathematician Oka, who proved much of the foundational theory for analytic sheaves.

Then one develops a number of technical analogues of properties of manifolds, in particular of products, compactness, and Hausdorffness. Since the Zariski topology is very coarse, the usual version of Hausdorffness almost always fails but there is an analogue of separation which usually holds. In making these constructions, mapping properties come to the fore, and are crucial even for finding the right definitions, so categorical thinking is essential.

A product is thus a space with projections such that morphisms to a product are equivalent to pairs of morphisms to the factors. There is no guarantee that all the points of the product will be given entirely by pairs of points of the factors.

Having defined products, Hausdorffness, now called separatedness, of V, is characterized by the property of the diagonal of VxV being closed.

For separated varieties, compactness of V, now called properness, means the projection VxW-->W is a closed map for all W.

REMARK:
It is occasionally useful to keep in mind, that some technically valuable varieties are not separated even in the generalized sense. I.e. one may be able to prove a theorem by relaxing the requirement of algebraic separation.

WARNING:
The process of recreating within ring theory all the machinery of complex manifolds and algebraic topology is very time consuming. If one sets out to master all these foundations before doing anything concrete, or without knowing their analogs in classical geometry, it is easy to get discouraged and quit.

VI. COHOMOLOGY

To take full advantage of methods of algebraic topology one wants to define invariants which help distinguish between different algebraic varieties, i.e. to measure when they are isomorphic, or when they embed in projective space, and if so, with what degree and in what dimension. One hopes to recover within algebra the rich structure that Riemann gave to plane curves using classical topology and complex analysis. Since the Zariski topology is so coarse, again one must use fresh imagination, applied to the information in the structure sheaf, to extract useful definitions of basic concepts like the genus, the cotangent bundle, differential forms, vector bundles, all in a purely algebraic sense.

This means one looks at "sheaf cohomology", i.e. cohomology theories in which more information is contained in the rings of coefficients than in the topology. This is essential since the Zariski topology is coarse, while the rings are richly structured. E.g. the genus of a smooth projective curve V over an algebraically closed field, is the rank of $H^1(V,O)$, where O is the structure sheaf. But for the constant sheaf Z of integers, $H^1(V,Z) = 0$, since Z is "flabby" in the Zariski topology, so the usual cohomology group from algebraic topology does not give the right answer.

The first definition of sheaf cohomology for algebraic varieties was given by Serre in the great paper Faiseaux Algebriques Coherent, where he used Cech cohomology with coefficients in "coherent" sheaves, a slight generalization of vector bundles. (They include cokernels of vector bundle maps, which are not always locally free where the bundle map drops rank. This is needed to have short exact sequences, a crucial aspect of cohomology.) Cech cohomology is analogous to simplicial or cellular homology, in that it is calculable in an elementary sense using the Cech simplices in the nerve of a suitable cover, but it can also be cumbersome for complicated varieties. Worse, for non coherent sheaves which also arise, the Cech cohomology sequence is no longer exact.

Other constructions of cohomology theories by resolutions ("derived functors"), e.g. by flabby sheaves or injective ones, have been given by Grothendieck and Godement, which always have exact cohomology sequences, but which then necessarily differ from the Cech groups, hence computing them poses new challenges. (Just as one computes the topological homology of a manifold from a cover by cells which are themselves contractible, hence are "acyclic" or have no homology, one also computes sheaf cohomology from a resolution by any acyclic sheaves - sheaves which themselves have trivial cohomology. This is the key property of flabby and injective sheaves.)

As in classical algebraic topology, no matter how abstract the definition of cohomology, it becomes somewhat computable, at least for experts, once a few basic exactness and vanishing properties are derived. A fundamental result is that affine schemes have trivial cohomology for all coherent sheaves. This makes it possible to calculate coherent Cech cohomology from any affine cover, without passing to the limit, e.g. to calculate the cohomology H*(P^n, O(d)) of line bundles from the standard affine cover of projective space. But once the affine vanishing property is proved for derived functor cohomology, it too allows computation of the groups H*(P^n, O(d)).

VII. SPECIAL TOPICS

It is hard to prove many deep theorems in the generality of arbitrary schemes. So having introduced the most general and flexible language, one often returns to the realm of more familiar varieties and tries to study them with the new tools. E.g. one may ask to classify all smooth irreducible curves over the complex numbers, or all surfaces. For instance, even much of Mumford's and Hartshorne's work deals with moduli of curves, even curves in projective space. The last collection of Grothendieck's conjectures in his "Esquisse d'un programme", intensely studied for the past 30+ years, concern what information is contained in the etale homotopy groups of curves over finite fields.

One can study the interplay between topology, analysis and algebra in higher dimensions as Riemann did for curves, and ask e.g. what restrictions exist on the topology of an algebraic variety. Hodge theory, the study of harmonic forms, plays a role here. Griffiths and Deligne have done much work advancing, and generalizing, this subject.

Instead of global questions, one can focus on local issues, e.g. singularities, the special collapsing behavior of varieties near points where they do not look like manifolds. Brieskorn said there are three key topics here: resolution, deformation, and monodromy. <u>Resolution</u> means removing singularities by a sort of topological surgery while staying in the same rational isomorphism class. <u>Deformation</u> means changing the complex structure by a different sort of surgery which allows the singular object to be the central fiber in a family of varieties whose union has a nice structure itself. Deformation leaves the algebraic invariants of a variety more nearly constant than does resolution. <u>Monodromy</u> means studying what happens to the topological or other subvarieties of a smooth fiber in a family, as we "go around" a singular fiber and return to the same smooth fiber.

E.g. if a given homology cycle on a smooth fiber is deformed onto other nearby smooth fibers, when it goes around the singular fiber and comes back to the original smooth fiber, it may have become a different cycle! I.e. if we view the homology groups on the smooth fibers as a vector bundle on the base space, sections of this bundle are multivalued and change values when we go around a singularity, just as a logarithm changes its value when we go around its singularity at the origin.

People who study particular algebraic varieties may look for ones that are more amenable to computation than very general ones, e.g. curves, special surfaces, or group varieties like abelian varieties. The latter is my area of specialization, especially abelian varieties arising from curves either as Jacobian varieties, which parametrize line bundles on curves, or as components of a splitting of Jacobians induced by an involution of a curve (Prym varieties).

Others study curves, surfaces and threefolds which occur in low degree in projective space such as curves in projective 3 space, or as double covers of the projective plane or of projective 3 space branched over hypersurfaces of low degree such as quartics. Dual to varieties of low dimension are those of low codimension, e.g. projective hypersurfaces, varieties defined by one homogeneous polynomial. Some study vector bundles on curves, or on projective space.

Some researchers following the lead of Riemann, Mumford, Grothendieck and others, examine how varieties vary in families. A beautiful class of objects called "moduli" varieties, are candidates for base spaces of "universal" families of varieties of a particular kind, the guiding case always being curves. A very active area is the computation of fundamental invariants of the moduli spaces M(g) of curves of genus g, and of their enhanced versions M(g,n), moduli of genus g curves with n marked points.

Another very rich source of accessible varieties is the class of "toric" varieties, ones constructed from combinatorial data linked to the exponents of monomials in the defining ideal, or more recently "spherical varieties".

VIII. PRERECQUISITES

To do algebraic geometry it obviously helps to know complex analysis, commutative algebra, algebraic topology, number theory, categories and functors, sheaf cohomology, harmonic analysis, group representations, differential manifolds,... even graphs, combinatorics, and coding theory! But to know all this before starting is hopeless, and one can begin studying the most special example one finds attractive, such as elliptic curves (curves of genus one), or rationally parametrizable surfaces such as quadric hypersurfaces, or complex tori, and use this study to motivate learning some more tools. This is a commonly recommended way to begin.