epsilon camp problem set 2013:

Well, camp is almost here, and the problems forum is winding down, at least for now. We have treated there, as you may have followed, solving both quadratic and cubic equations, rational root theorem, and applications to problems, then ideas of multiplicities of roots combined with the concept of crossing or not crossing the X axis at a root, with application to the concept of a tangent line, both as used by Descartes, Euclid, and ultimately Newton.

We found slope formulas for all polynomials, and even negative power functions of X. We observed the fundamental impoprtance of the intermediate value theorem and Rolle's theorem. This constitutes a small conceptual introduction to theory of equations and differential calculus.

Then we used the ideas learned about tangent lines, to solve some optimization problems, and to give an inductive proof of the rule of signs stated by Descartes and usually named for him, but actually proved earlier it seems by Abbe' de Gua.

Next we took an excursion to the "LaGrange interpolation formula" which allows one to write a unique polynomial of degree at most n and having any desired values at any n+1 different points. This shows that the graph of a polynomial of degree n is completely determined by any n+1 distinct points on the graph. Conversely, given any n+1 points in the plane, with different X coordinates, there is a unique graph of a polynomial of degree n passing through all of them.

This generalizes the familiar fact that through any two distinct points there passes a unique line (although for lines the 2 points do not need to have distinct X coordinates. Perhaps if we used more general equations in both X and Y we could also partially eliminate this in the case of curves of higher degree. No I don't think so, but this leads us to the theory of plane curves, a topic for some other discussion.)

Finally, we began a discussion of the familiar formula for summing up the first n integers, and generalized it to summing the first n squares, and beyond. We learned, without full detail, how to see inductively that the formula for the sum of the first n kth powers, i.e. for 1^k + 2^k + ... +n^k, always has form n^(k+1)/(k+1) + terms of degree lower than k+1.

With just these partial formulas for such sums of powers, we are able to find the area under the graph of any monomial Y = X^k, between X=0 and X=any X. The "moving area function" for that region was X^(k+1)/(k+1), mimicking the lead term of the formula for the sum of the first n kth powers. By subtracting we get the area between any two points X=a and X=b, and by additivity and linearity of area, this allows us to evaluate the area under the graph of any polynomial.

Moreover it shows us that the derivative (slope of tangent line) for the area formula, equals the height of the original graph whose area is being sought. I.e. the slope of the graph of the area function is the height of the original function, or the derivative of the area function is the height function. This is called one of the fundamental theorems of calculus, and we have proved it for polynomials.

The final topic, generalizes this area calculation to a calculation of volumes. The open question there is what is the derivative of the volume function? I.e. how is the derivative of the moving volume function for a solid related to the geometry of that solid? Or, as we move slightly to the right on our solid, i.e. in the X direction, by what factor does the volume increase?

A hint was that for areas, the derivative of the area function was the size (length) of the leading edge of the moving region whose area was being measured, i.e. the length of the part of that region having X coordinate exactly equal to X. So for volume presumably it has something to do with the geometry of the part of the solid having X coordinate equal to X also.

This has been enormous fun for me, and I want to thank everyone who participated and/or looked on. Have fun at camp next week!

Problems, hints, and related theory.

These three algebra problems are from the book Elements of Algebra by Euler, the great 18th century mathematician.

1. Find a number such that if we multiply half of it by a third of it, and to the product add half of it again, the result will be 30.

2. Find two numbers, the one being double the other, and such that, if we add their sum to their product, we obtain 90.

3. A father leaves to his four sons $8,600 and , according to the will, the share of the eldest is to be double that of the second, minus $100; the second is to receive three times as much as the third, minus $200; and the third is to receive four times as much as the fourth, minus $300. What are the respective portions of these four sons?

Another problem from Euler, and more:

4. A man bought a horse and then sold it for 119 crowns. His gain on this sale, as a percent of the purchase price, equaled the numerical purchase price. What was that purchase price?

5. A. If r,s are the solutions to the quadratic equation, X^2 - bX + c = 0, find formulas for r+s, and rs, and (r-s)^2, all in terms of b and c.

HINTS: a) Use the quadratic formula to express the solutions r,s, in terms of b,c and then compute.

OR b) If X = r,s are the solutions, how would the equation factor? E.g. how does the equation X^2-7X+10 = 0 factor, with solutions 2,5?

B. i) Write down a quadratic equation whose two solutions r,s have sum 22 and product 85. What are the coefficients? Is this pattern always true?

iii) compute (r-s)^2 and (r-s) for this equation, preferably without solving the quadratic equation first.

iii) Now that you know both r+s and r-s, can you use them to find r and s? Does the expression for r and s look familiar?

If we want to find square roots of negative numbers, we only need to find a square root of -1. Let's imagine it is a number we may not have met before but that exists in our imaginations, and call it i for "imagine". Then also -i is another square root of -1. Now we have all we need, because then e.g. sqrt (-16) = ±4i, and sqrt(-49) = ±7i, etc...

6. i) Write an equation whose roots r,s add up to 4 and multiply out to 5.

ii) find (r-s)^2, the square of the difference of the roots.

iii) write down the two roots r,s, (using i (=sqrt(-1)), if necessary).

7. The first recorded instance I know of solving quadratic equations, is in the book Arithmetica by Diophantus, but much of his book apparently concerned linear equations. When he died his tombstone was reportedly inscribed as follows (in Greek):

“This tomb holds Diophantus. Ah, what a marvel. And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled him the light of marriage after a[nother] seventh, and in the fifth year after his marriage He granted him a son. Alas! Late begotten and miserable child, when he had reached the measure of half his father’s (full) life (span), the chill grave took him. After consoling his grief by this science of numbers for four years, he (Diophantus) reached the end of his life.”

What was the length of Diophantus’ life?

Now I want to ask another somewhat general question. We "know" that whenever we can factor a quadratic equation into linear factors, say X^2 - bX + c = (X-r)(X-s), that then the "roots", i.e. the solutions of
X^2 - bX + c = 0, must be X=r and X=s.

#8a. How do we know this? I.e. why do these solutions work? and why do no other numbers work?

i.e. assume say that b,c,r,s are any rational numbers such that X^2 -bX +c = (X-r)(X-s).

Then explain why setting X=r makes X^2 -bX +c = 0, and also setting X=s does so, and explain why there cannot be any other rational numbers t, such that setting X=t makes X^2 -bX +c = 0.

And does this solution method, by factoring, always work? I.e.
#8b. If we assume that X=r is a root of X^2 -bX + c, i.e. if we assume that setting X=r makes X^2 -bX + c = 0, then is it guaranteed that (X-r) is a factor of X^2 -bX + c?

I.e. do we then always have a factorization of form X^2 -bX +c = (X-r)(X-s), for some s?

Assume first that b,c are rational numbers and r is a rational root, and see if you can show a rational s exists such that X^2 -bX +c = (X-r)(X-s).

I.e. show that if r is a rational root, then (X-r) is a factor "over the rationals".

[That means show that not only the factor (X-r) contains only rational numbers, but that all factors involved contain only rational numbers. In general the more numbers we allow, the easier it may be to find a factorization. Thus a factorization “over the reals” is one in which all factors involve only real numbers, while a factorization over the complex numbers allows the use of non real numbers in some factors. E.g. $X^2 + 1$ cannot be factored over the reals, but it can be factored as $(X+i)(X-i)$ over the complexes.]

If you can do that, try assuming b,c are integers and r is an integer root, and then show that the rational number s, i.e. the other root, will also be an integer in that case.

What do you think happens if r is a real (possibly irrational) root? or a complex root? Will (X-r) be a factor "over the reals"? or "over the complex numbers"?

By the way here is Diophantus' solution to finding two numbers r,s whose sum is 22 and whose product is 85.

He knows he always wants to find their difference r-s, so he starts by putting x = r-s.

Since 22 = r+s, then 22+x = (r+s) + (r-s) = 2r, and 22-x = (r+s) - (r-s) = 2s.

Then he knows that the product rs = 85, so (22 + x)(22-x) = (2r)(2s) = 4rs = 4(85) = 340.

This equation for the difference r-s = x, becomes (22)(22) - x^2 = 484-x^2 = 340, or x^2 = 484-340 = 144.

Thus the difference x = r-s = 12, so then 2r and 2s are 22+12 = 34 and 22=12 = 10.

You can see the innards of the quadratic formula in there, but he has made the computation easier, by focusing only on computing the discriminant (r-s)^2 = b^2-4c = 484-340 = 144.

Solutions to general cubics had to wait over a thousand more years, and we will see gradually how this was done.

#9. Diophantus seems to have solved only one cubic equation in his book. He did not have a general formula so he had to use one of the other methods. See if you can solve this cubic, apparently the only one solved in Diophantus' book: $X^2 + 2X + 3 = X^3 + 3X - 3X^2 - 1$.

(Hint: He started by "collecting terms" and also transforming until all coefficients are positive on both sides, but that is not necessary.)

10. Assume that b,c are integers and that
$r = (1/2)(b + \mathrm{sqrt}(b^2-4c))$,
and $s = (1/2)(b - \mathrm{sqrt}(b^2 - 4c))$.

Then show that:
i) $r+s = b$, and $rs = c$.

ii) $(X-r)(X-s) = X^2 - bX + c$.

iii) r,s are “roots” of $X^2 - bX + c$, i.e. setting X = either r or s, makes $X^2 - bX + c = 0$.

iv) there are no other rational, real, or complex roots of $X^2 - bX + c$, other than r,s.

(Hint: the product of two rational, real, or complex numbers cannot be zero, unless one of the two numbers is zero.)

#11. Assume that b,c, are integers, and that one root r of X^2-bX+c is an integer. then prove that the only other root s is also an integer. I.e. prove that if there is one integer r such that X=r makes $X^2-bX+c = 0$, then all roots of this equation are integers.
#12. Prove that a quadratic equation $X^2 - bX + c = 0$, where b,c are integers, cannot have more than two distinct solutions.

I've been thinking about the relation between remainder after division by (X-r), and the trick of casting out 3's to see if a number is divisible by 3. I want to try to explain them as if they are almost the same idea.

If we want to divide a number by 3, we just subtract off copies of 3, until we can't do so any more, because we get something less than 3. Then the number of times we subtracted off 3's is the (partial) "quotient" and the number smaller than 3 that remains, is the “remainder”.

if all we want to know is whether the original number is evenly divisible by 3, then we don't need the quotient, just the remainder, i.e. we just want to know whether or not the remainder is zero. so what we do is throw away 3's until we don't have any 3's left.

Another way to say we throw away 3's, is to "set 3's equal to zero". E.g. starting from 10, we see that 10 = 9+1, and 9 is an even multiple of 3, so throwing away 3's from 10 leaves 1.

Now 100 = 99 + 1, and 99 is also a multiple of 3, so throwing away threes from 100 also leaves 1. Another way to look at it is that 100 = 10^2 and thus (10)^2 = (9+1)^2 = 9^2 + 2.9 + 1^2, so throwing away 3's leaves 1^2.

Similarly, throwing away 3's from 1000 = 10^3 leaves 1^3 = 1.

So if we want to know whether a number like 47298 is divisible by 3, we can rewrite it as

4(10,000) + 7(1000) + 2(100) + 9(10) + 8

= 4(9,999+1) +7(999 + 1) + 2(99+1) +9(9+1) + 8,

so throwing away 3's, i.e. setting multiples of 3 equal to 0, leaves 4(1) + 7 (1) + 2(1) + 9(1) + 8 = 21, and throwing away more 3's leaves 0.

Hence if we throw away all multiples of 3 in 47298, we get 0, so it is divisible by 3.

Intuitively, if we set 3 = 0, then 10 becomes 1, because we must also set 9 = 3(3) = 3(0) = 0, and since 10 = 9+1, and we have set 9 = 0, then 10 becomes 1. Similarly 100 becomes 1, 1000 becomes 1, and so each number becomes just the sum of its digits.

So 47,298, after setting 3 equal to zero, becomes 4+7+2+9+8 = 30, which becomes zero. But setting 3's equal to zero, leaves the remainder after division by 3, so 47,298 is divisible by 3, i.e. it has remainder zero after division by 3.

We can do something similar for division by 7. I.e. when we set 7 ≈ 0 (let’s write ≈ for setting equal to zero) then 10 becomes 7+3 ≈ 3, and 100 becomes 10^2 ≈ 3^2 = 9 ≈ 2.

So 846 = 8(100) + 4(10) + 6 becomes 8(2) + 4(3) + 6 = 34 = 28+6 ≈ 6, after throwing away 7's.

So this number has remainder 6 after division by 7 and thus is not divisible by 7.

What is 1000 after throwing away 7"s? Since $1000 = 10^3$, and 10 becomes 3, 1000 should become $3^3 = 27$, or 6.

So what is the remainder of 9175 after division by 7? By this computation it should be $9(6) + 1(2) + 7(3) + 5 \approx 82 \approx 77 + 5 \approx 5$. So not divisible by 7.

Is this right? Let's see... $9175 = 7000+2175 = 7000 + 2100 + 75$
$= 7000 + 2100 + 70 + 5 \approx 5$. So it does check.

What about division by 11? Notice now that $10 = 11 -1$, so 10 becomes -1, and $100 = 99+1$, $= 9(11) + 1$, so 100 becomes +1.

More easily, $1000 = 10^3$ becomes $(-1)^3 = -1$, and $10{,}000 = 10^4$ becomes $(-1)^4 = 1$, and so on.

So after division by 11, the number 4791 becomes $4(-1)+7(1)+9(-1)+1 = 8-13 = -5 \approx 6$, so the remainder after division by 11 is 6, and 4791 is not divisible by 11.

Let’s check that too. $4791 = 4400 + 391 = 11(400) + 11(30) + 61 \approx 61$
$\approx 55 + 6 \approx 6$. (I made several errors at first, doing it this way.)

Ex. Is 8943762 divisible by 7? 11? 3? what are the remainders?

Next we explain what this has to do with checking divisibility of polynomials by (X-r).

Suppose we want the remainder of a polynomial after division by (X-r).
By analogy with the case of division by 3, we want to set (X-r) equal to zero and see what remains,
but here is the whole point: setting $X -r = 0$ is the same as setting $X = r$.

And setting $X = r$ means substituting $X = r$ in the polynomial.

Thus the remainder of aX^2+bX+c after division by (X-r) is the same as the result of setting $X =r$, and the same as setting $X=r$, namely ar^2+br+c.

Thus a polynomial is divisible by (X-r) if and only if we get zero when we set X=r in that polynomial.

i.e. (X-r) is a factor, if and only if r is a root.

This fundamental result is often called the “factor theorem” in algebra books.

in particular, since 5 is a root of X^2 - 14X +45, then (X-5) must be a factor. indeed X -9 seems to be the other factor.

In fact the two phenomena are related further since a decimal integer is just a polynomial, with coefficients between 0 and 9, and with X set equal to 10.

I.e. 49672 = 4(10)^4 + 9(10)^3 + 6(10)^2 + 7(10) + 2, is just
4X^4 +9X^3 + 6X^2 + 7X + 2, with X = 10.

So if X-r divides a polynomial, then 10-r should divide the corresponding decimal.

Since all the coefficients of a decimal are positive, a positive r will never be a root, so X-r will never divide such a polynomial.

But e.g. if r= -1, then X-(-1) = X+1 divides a polynomial if and only if X = -1 is a root.
E.g. X = -1 is a root of X^3+X^2+X+1, so X+1 must divide it. Thus also 10+1 = 11 must divide the corresponding
decimal 10^3+10^2+10+1 = 1111. Indeed this number passes the test for division by 11 we saw above, which was precisely setting each power of 10 equal to the corresponding power of -1.

In fact 1111 = 11(101), I believe, just from “eyeballing” it.

Lets try this on division by 13 = 10+3. I want a decimal that gives a polynomial that has -3 as a root.

But setting X = -3 in the polynomial will be the same as setting 10 = -3 in the decimal, or setting 13 = 0.

So the tests really are the same.

E.g. X^2+6X+9 has $X= -3$ as a root, so is divisible by $X+3$. And similarly, 169 becomes $(-3)^2 + 6(-3) + 9 = 0$ when I set $10 = -3$, i.e. set $13 = 0$, so 169 is divisible by 13.

Notice the test for factoring using polynomials gives something new for numbers. I.e. if a polynomial, like X^2+2X+1 equals zero when we set $X=-1$, it means it i divisible by $X+1$. This means that if we think of the polynomial as a number in "base X", then the number it represents is divisible by one more than the base.

So the coefficients of this polynomial, namely "121", not only represent a number divisible by 11 in base ten, but in base 12, it represents the base ten number $1(12)^2 + 2(12) + 1 = 169$, which is divisible by (the base ten number) 13, and in base 15, "121" represents the (base ten) number $1(15)^2 + 2(15) + 1 = 256$, which is divisible by 16 .

#13. Here is another fun little problem from Euler:

Again, Suppose twenty persons, men and women, go to a tavern; the men spend 24 shillings, and the women as much : but it is found that the men have spent 1 shilling each more than the women. Required the number of men and women separately ?

Next I want give Euler's explanation of how to solve cubic equations. first he shows that any cubic equation can be transformed by a trick to change the cubic into one with no X^2 term. So it is only necessary to be able to solve cubics like this one:
$X^3 = pX + q$. E.g. suppose that we have $X^3 = 9X + 28$.

Then Euler explains that to solve this all we need to do is find two numbers u,v such that $3uv = 9$ and $u^3 + v^3 = 28$. Then $X = u+v$ will solve the cubic.

14. See if you can use Euler's method to solve $X^3 = 9X + 28$.

(Essentially this solution method was found apparently by Scipio del Ferro, and later Tartaglia, who explained it to Girolamo Cardano, who eventually published it. It is often called "Cardano's method".)

15. Try this one as well: $X^3 = -18X + 19$.

Solving cubics used to be taught in elementary algebra books, but when I went to high school, it was no longer done, and I think is not commonly done now. This is another reason to prefer the great old algebra books like those of Euler and LaGrange.

16. Show this method can be used on all cubics of form X^3 = pX + q.

I.e. i) show that one can always find numbers u,v such that u^3+v^3 = q, and 3uv = p.

(Hint: let A=u^3 and B=v^3. Then you know that A+B = q, and AB = p^3/27; why? Then show how to find A and B, and then tell how to find u and v.)

ii) Show that if X = u+v and if 3uv = p and u^3+v^3 = q, then X^3 = pX +q.

The whole point of understanding quadratic equations is this. Write them like X^2-bX+c = 0, and then b is always the sum of the solutions and c is always their product.

Vice versa, whenever you are looking for two numbers and you already know their sum and their product, then you can always find the numbers as the solutions of a quadratic equation.

The whole point of solving cubics is this:
First we know from studying quadratics that we can always find two numbers whose sum and product are known,.
Next Euler shows that to solve the cubic X^3 = pX + q, all you need is two numbers u,v such that u^3 +v^3 = q, and 3uv = p.

So we just need to find u and v. But it turns out we already know how to find their cubes, u^3 and v^3.

I.e. what do we know about the cubes of u,v? we know their sum u^3 + v^3 = q, and we know their product, since 3uv = p, so 27 u^3v^3 =p^3, hence u^3v^3 = p^3/27.

So since we know the sum and product of u^3 and v^3, we can find u^3 and v^3 by solving a quadratic! Then we can take cube roots to find u and v.

I.e. if r is a root of the quadratic t^2 - qt + (p^3/27), set u = cuberoot(r) and v = p/3u, and then u + v = X solves the cubic!

Euler mentions another interesting fact about solving equations by educated guessing, in passing. It is true that if we are only looking for rational solutions of a polynomial like $X^4 + 6X^3 + X - 12 = 0$, we only need to look at integer factors of the constant term, namely factors of 12. But we have to look at both positive and negative factors. Since there are a lot of those, ± 1, ± 2, ± 3, ± 4, ± 6, ± 12, trying them all takes a lot of time.

But Euler remarked that "we also know" there cannot be more positive roots than there are changes of sign in the sequence of coefficients. Since here there is only one sign change, from $+X$ to -12, as soon as we find one positive root, we can stop looking at the positive factors.

I had never learned this algebra fact, attributed to Descartes, and even earlier to M. l'Abbe' Gua, but I think I understand it now.

Problem #17: Explain why (prove) a polynomial of degree one or two cannot have more positive roots than it has sign changes in its sequence of coefficients. Check it out first to see if you believe it.

Challenge: Explain why this is true for polynomials of every degree. (Hint: one way is to use induction.)

#18. If two rational numbers are not both integers, can both their sum and their product be integers? If so give an example, if not give an argument. Explain why this question is the same as asking whether a quadratic polynomial of form $X^2 - bX + c = 0$ can have rational roots that are not integers, if the coefficients b,c are integers. (Hint: This may help solve it.) [e.g. note that 2/3 and 3/2 are not integers and while their product is 1, an integer, their sum is $4/6 + 9/6 = 13/6$, not an integer.]

#19. Solve $X^3 = 36X + 91$.

Recall that $X = u+v$ solves a cubic of form $X^3 = pX + q$, if $u^3+v^3 = q$ and $3uv = p$.

one more problem from Euler:
#20. Several persons form a partnership, and establish a certain capital, to which each contributes ten times as many pounds as there are persons in the company; they gain 6 plus the number of partners per cent; and the

whole profit is 392 pounds; required (to determine) how many partners there are.

Write an equation for that number, and try to solve it, bearing in mind the solution is a positive integer.

wow! i just convinced myself of how useless the cubic formula can be with some examples;
try this little cubic: $X^3 + X - 2 = 0$. How hard is it to guess a solution? but just try using the formula!!

gee, here's an even simpler one, $X^3 = X$. I can get 1 and -1 from the formula, but not 0 so far. Try it. Maybe $X^3 = 3X$, will be easier.

Here is an attempt to show that any rational solution to a "monic" equation (lead coefficient = 1), with all integer coefficients, must be an integer.

Let the equation be $X^n + aX^{(n-1)} + bX^{(n-2)} + \dots + dX + e$, all coefficients integers.

Assume p/q is a rational root in" lowest form", i.e. in which no prime factor of q divides p. Then plug in $X = p/q$, and multiply out the denominators by multiplying through by q^n.

We get $p^n + ap^{(n-1)}.q + bp^{(n-2)}.q^2 + \dots + dpq^{(n-1)} + eq^n = 0$. Since q divides every term but the first, it also divides that term, so q divides p^n. But q had no prime common factors with p. I claim then it cannot have any in common with p^n either, so the only possibility is for $q = 1$, i.e. the rational root is an integer.

To complete this argument we need to know that if a prime does not divide an integer then it also cannot divide any power of that integer. does that seem plausible? any ideas for proving it?

A more general result says that a rational root p/q (in lowest form) of a polynomial with integer coefficients, $aX^n + bX^{(n-1)} + \dots + dX + e$, must have p a factor of e, and q a factor of a.

Now should we try to prove that criterion?? (yes.)

Here is a little fact I myself found hard to prove but Gauss proved it. This is not that surprising of course. (reference: Disquisitiones Arithmeticae, by Gauss.)

Lemma: if p is a prime number, i.e. p is a positive integer greater than 1 and not divisible by any smaller positive integer except 1, then no multiple of p can be a product of two positive numbers both smaller than p and greater than 1.

I.e. assume p is prime, then prove it is not possible to have $np = ab$, where $1 < a, b < p$, and n is an integer.

Gauss's proof involves fixing a, choosing b as small as possible, and getting a contradiction.

The Lemma does follow from the theorem on unique factorization of integers > 1 into primes. We want to see if we can prove it without that, i.e. allowing the (indeed false) possibility that there might be 4 different primes a,b,p,n with $ab = pn$.

Before Gauss, most people seem to have assumed unique prime factorization without proof, so he was trying to prove it using the Lemma above.
Here is another essentially equivalent version of the Lemma:
if a prime p divides a product ab, then it must divide one of the factors a or b.
(a,b, integers, and to say one integer "divides" another means the quotient is also an integer.)

Gauss has a clever proof of the Lemma by contradiction, but there is a more usual argument using the fact that the "greatest common (integer) divisor" of two integers, can be written as a sum of (possibly negative) integer multiples of the two integers.

E.g. the gcd of 8 and 11 is 1, and $1 = 3(11) - 4(8)$.

This fact was due to Euclid, and can be used to deduce unique prime factorization, but I did not find that deduction in Euclid.

Unique prime factorization is also true for complex integers of form $a + b\,i$, with a,b, integers, but it fails for some similar complex systems, like all numbers of form a + b.sqrt(-5) with a,b, integers, since there (1+sqrt(-5))(1-sqrt(-5)) = 6 = 2(3), gives two different factorizations.

You might think that since $(1+3i)(1-3i) = 10 = 2(5)$, that complex integers also fail unique prime factorization, but 2 and 5 are not primes in that system, since $2 = (1+i)(1-i)$ and $5 = (1+2i)(1-2i)$.

Let's separate out two different facts. If we know that every integer > 1 has one and only one factorization into positive primes, apart from the ordering of the factors, then it follows that a rational root p/q in lowest terms, of a polynomial
$aX^n + bX^{(n-1)} +.....+dX + e$, with integer coefficients, must have p dividing e, and q dividing a.

This is called the rational root theorem, and it holds whenever unique prime factorization holds.

To review the proof, substitute p/q for X and multiply out by q^n, getting
$ap^n + bp^{(n-1)}q +....+dpq^{(n-1)} + eq^n = 0$.

Then p divides every term but the last hence it also divides the last. But p/q is in lowest terms, so p has no prime factors in common with q.

Since p divides eq^n, then all prime factors of p must be among the prime factors of e, so p divides e.

Similarly, q divides ap^n, and since no prime factors of q are factors of p, q must divide a.
A nice corollary of this theorem is the fact that there are no rational square roots of integers that are not squares of other integers.

I.e. let n be a positive integer. then a square root of n is a solution of $X^2 - n = 0$, which has integer coefficients. Thus by the rational root theorem, the only possible rational square roots of n are among the integer factors of n. In particular if n is prime it has no rational square root, so its square root must be irrational.

So we are interested in why integers do satisfy "unique prime factorization". The fundamental fact needed for that proof is that a prime cannot divide a product of two integers unless it divides one of the factors. So that one fact needs to be proved without using unique prime factorization.
Here is the argument of Gauss:

assume $0 < a < p$, and there is some b also with $0<b<p$ such that p divides ab. Choose the smallest possible such b.

Then p being prime, is not divisible by b, so $p = nb + r$, with some remainder r with $0 < r < b$.

now $ar = a(p-nb) = ap - nab$, and both terms on the right are divisible by p (since ab is so by hypothesis, and ap has p as a visible factor), hence so is the left side, i.e. ar is also divisible by p.

But this says r is a number smaller than b, that also multiplies a into a multiple of p, contradicting the choice of b as the smallest such number (integer). What do you think of this?

Can you use this to prove that a prime p that divides a product ab of any two integers must divide either a or b?

Let's see how useful that rational roots theorem is for guessing rational roots.

See if you can find a rational root of this guy: $81X^4 - 81X = 2212$.

As we have seen, it helps to get a rough idea of how big it should be. Notice also there is no integer solution, since if X were an integer, the left side would be an integer divisible by 81, whereas the right side is an integer not divisible even by 3. That also contains a hint as to the nature of the solution.

The rule of signs also tells us there is at most one positive root (because only one sign change in $X^4 - 81X - 2212$), and (substituting $-X$ for X and checking sign changes) also at most one negative root. so although this quartic has 4 (real or complex) roots, apparently two are imaginary.

What if, instead of asking where $81X^4 - 81X - 2212$ equals zero, we wanted to know where it has as small a value as possible? Can you estimate where this would occur? Why does there have to be such a place at all?

See what you think of this argument: if we look where $x=0$, we have value -2212. If we look at negative values of x, then we are adding to -2212, the values of $81X^4$ and $-81X$. But since X is negative both of these terms are positive, so we get something greater than -2212 for every negative value of X.

Now if we look at positive values of X, as soon as X^4 is greater than X, i.e. as soon as X > 1, we have the first two terms 81X^4 -81X again positive, so the values are again greater than -2212.
Thus the values of this polynomial are greater than -2212 whenever X > 1 and whenever X < 0. So the smallest value should occur somewhere between X =0 and X = 1, and that smallest value should be less than -2212. And there should be such a value, since the part of the graph between those points is a finite continuous curve joining the points (0,-2212) and (1,-2212). So there should be a lowest point of the graph somewhere between these two points.

But let's start as we always should do, with a simpler case than a quartic. We will learn more if we begin where we are familiar, with quadratics.

#21. For what value of X does the polynomial Y = X^2 have the smallest value of Y? and what is that smallest value of Y?

#22. For what value of X does the polynomial Y = (X-2)^2 + 7 have the smallest value of Y, and what is that smallest value of Y?

#23. For what value of X does the polynomial Y = X^2 - 6X -12 have the smallest value of Y, and what is that smallest value of Y?

#24. challenge: in each case, if Y = c is the smallest value of Y attained by the expression given, then how many times does the line Y = c, meet the graph of the given quadratic, (i.e. of Y = X^2, or Y = (X-2)^2 +7, or Y = X^2-6X -12)?

I would like to ask you now to think about the connection between geometry and the number of roots of equations. We "know" that a polynomial of degree n should have at most n roots, in an algebraic sense, because that is the most linear factors it can have. I.e. if a1,...an are all roots, then the polynomial is divisible by the product (X-a1)(X-a2)....(X-an), which already has degree n in X when multiplied out.

Of course some of those roots may be complex.
There is a geometric side at least to the number of real roots, since a real root occurs precisely where the graph meets the X axis. I claim this makes it possible to "see" geometrically why a polynomial of odd degree should have an odd number of real roots, and that a polynomial of even degree should have an even number of real roots.

Here is a basic principle: a continuous plane curve that starts out on one side of a given line at point A, and ends up on the opposite side of that same line at point B, must cross that line an odd number of times between the points A and B.

Similarly, a curve that starts out at A and ends up at B, with both points on the same side of a given line, crosses that line an even number of times between points A and B, (possibly zero times, an even number).

Do you believe that? Can you think of any counter examples or special cases?

What if you used the word "meets" the line, instead of "crosses" it? Would that allow any counterexamples or special cases?

Remember that a polynomial can have a multiple root, corresponding to a multiple factor, such as the root X=1, for the polynomial (X-1)^2.(X-3) = X^3 - 5X^2 +7X - 3.

#25. i) What happens to the graph of Y = X^3 -5X^2 +7X -3, at the point X=1? Does it meet the X axis? Does it cross it? What about at the point X=3?

ii) How can you tell geometrically that X=1 is a "double root" in the algebraic sense?
iii) at which roots does the value of Y change sign? i.e. does it change sign between X=0 and X=2? what about between X=2 and X=3?

Now we are going to learn about the "derivative". This is where many of the following questions are headed. So what is the derivative? Of course some of us know it represents the slope of the tangent line, so we are going to be investigating the concept of a tangent line, connecting up Euclid's idea of it with the algebraic approach of Descartes and Fermat, and Newton.

The reason Y=X^2 has slope zero at (0,0) is the absence of the X term, the “linear term”, which gives the slope there.

By translating the origin to X = 3, for the same reason Y=(X-3)^2 -21 has slope zero at X=3.

I.e. by “completing the square” for the quadratic $X^2 - 6X - 14$, via expanding in powers of $(X-3)$, you eliminate the linear term, i.e. the $(X-3)$ term.

If we translate $X^2 - 6X - 12$, to $X=a$, we get $[(X-a)+a]^2 - 6[(X-a)+a] - 12$

$= (X-a)^2 + 2a(X-a) + a^2 - 6(X-a) - 6a - 12$

$= (X-a)^2 + (2a - 6)(X-a) + a^2 - 6a - 12$, and this has no linear term, i.e. no $(X-a)$ term, when $2a-6 = 0$, i.e. for $a = 3$.

So to find a point where a polynomial graph has slope zero, we could try to find a point where the translate has no linear term.

What if we translate $81X^4$ to $X=a$? I.e. what if we substitute $X = [(X-a)+a]$ and multiply out?
Then what do we get for $81X^4$? What is the linear term? i.e. the $(X-a)$ term?

Then what is the linear, or $(X-a)$- term, for $81X^4 - 81X - 2212$, when we set $X = [(X-a)+a]$? and multiply out?

When is that term zero? i.e. for what value of a? Does that look familiar?

Finally, what is the $(X-a)$ term for X^n when you set $X = [(X-a)+a]$ and multiply out?

What if you transform a polynomial say to $X=5$, and it comes out looking like this: $(X-5)^5 - 7(X-5)^4 + 14$.

What can you say about the behavior of the graph near the point $X=5$? E.g. is it a local max? a local min? neither? What is the slope of the tangent line there?

And what if it comes out like this:
$(X-5)^5 - 7(X-5)^4 + 3(X-5)^3$? Then what is the nature of the graph near $X=5$? the slope of the tangent line?

Can you read off all this information without making any computations, i.e. without taking any derivatives?

Introduction to tangent lines

Since not everyone has studied tangent lines and derivatives, let's try to put everyone more on the same footing by giving an elementary introduction to that topic. There is no need to know anything about calculus or derivatives to follow this discussion.

Tangent lines are defined for circles in Euclid. Some of us may have heard that a tangent line to a circle is a line that meets the circle only once, and while this is true, it is not Euclid's definition, and it is not much use on most other curves.

Euclid defined a tangent line to a circle as a line that meets the circle but does not "cut" the circle, which seems to mean the line does not cut across the circle from one side to the other. So a tangent line is a line that meets a circle and stays on one side of the circle. This definition makes sense also for many other curves.

Look at the parabola with equation Y = 2+ 3X + 4X^2. This curve contains the point (0,2) and I claim at that point the tangent line is the line with equation Y = 2+3X, i.e. just the "linear part" of the equation of the parabola. To check it, let's observe that this line does meet the parabola at (0,2), since when X=0, the equation for the line equals 2 + 0 = 2, and that for the parabola equals 2+0+0^2 = 2. Furthermore, when X ≠ 0, then 4X^2 is positive, so for every other value of X, the height of the corresponding point of the parabola is higher by 4X^2 > 0.

Thus the parabola stays above the line everywhere except where it meets it at (0,2). In particular the line stays below the parabola, so does not cross it. Thus this line meets Euclid's definition of a tangent line. We say the slope of the parabola at the point (0,2) equals the slope of its tangent line, which here is 3.

Notice that this tangent line does indeed meet the parabola only once, but in fact the line X=0 also meets the parabola only once, at (0,2). That vertical line however crosses the parabola from below it to above it, hence does not meet Euclid's criterion for a tangent line.

Now how do we find the tangent line to this parabola at another point? Easy, we just translate, from X=0 to X=any other point. E.g. at X=2, we re expand the equation, replacing X by (X-2)+2, getting

2 + 3X +4X^2 = 2 + 3[(X-2)+2] +4[(X-2)+2]^2

= 2 + 3(X-2) + 3(2) + 4(X-2)^2 + 4[4(X-2)] + 4(4), and collect terms, getting

$= 24 + 19(X-2) + 4(X-2)^2$.

Thus the tangent line should be the line $Y = 24 + 19(X-2)$, which meets the parabola at (2,24) and everywhere else lies below it. The slope of this line and hence of the parabola at (2,24) is thus 19.

Let's find the tangent line to our parabola at a general point with $X=a$.
We just expand $2 + 3[(X-a)+a] + 4[(X-a)+a]^2$ and collect terms.

This gives $(2 + 3a +4 a^2) + (3 +8a)(X-a) + 4(X-a)^2$.

Thus the tangent line is $(3+8a)(X-a)$ and has slope $3 + 8a$.

In particular, when $a = 2$, we do get 19 for the slope as obtained before.

Then to find the lowest point on this parabola, we want the tangent line to be horizontal, with slope zero so that we will have $c + d(X-a)^2$ and no linear term at all. I.e. then setting $X=a$ will give value c, and any other value of X will add the positive term $d(X-a)^2$.

For this to happen we just set the slope $3+8a = 0$ and solve, getting $a = -3/8$.

#26: if $Y = 9 - 6X + 2X^2$, find the tangent line and slope at the point (0,9), and also at the point (1,5). Find the lowest point of this parabola.

Now I want to look at a general quadratic, and we may get confused because I am used to using the letter a also for the lead coefficient of a quadratic equation. So take the quadratic to be $Y = aX^2 +bX + c$, and take the point now to be $X=p$.

#27: Show the slope of the tangent line to this parabola at $X=p$, is equal to $2ap+b$. Thus at a point X, the slope is $2aX+b$.

#28: Find the highest point on the parabola $Y = aX^2+bX +c$, (for $a < 0$).

Here is a typical "calculus" problem:
29: A rectangular garden is to be constructed with one side open along a river, and the other three sides enclosed by a wire fence. If 800 meters of

wire fencing are available, what is the area of the largest such garden one can build?
Hint: write a quadratic formula for the area, and find the highest point on the corresponding parabola. Does the answer make sense to you? How could you have guessed it?

30: Find the linear term at $X=a$, of X^n, i.e. find the linear term after re - expanding in powers of $(X-a)$.
Hint: The binomial theorem should help, and you only need one term of it. It is wise to start with X^3, and X^4, to get the idea.

Tangent lines viewed the "modern" way:
There is one special case of higher degree curves where Euclid's definition of tangent lines for circles does not quite work. He said a tangent line should not cut across a circle, and that is true also for other curves, at least at all points where a curve is "convex" or "concave", such as a maximum or minimum point, but for higher degree curves there can be special points where the curve changes from convex to concave, i.e. "flex" points. At these points there is no line that stays entirely on one side of the curve.

E.g. at the point $(0,0)$ on the curve $Y = X^3$, the linear term is $Y = 0$, so the X axis should be the tangent line, but the X axis goes from one side of this curve to the other at $(0,0)$. The problem is the absence of the X^2 term.
In general here is what can happen: we can have a polynomial in X, and a point a such that
when we expand in powers of $(X-a)$ we get no quadratic term, i.e.
$f(X) = c + b(X-a) + (X-a)^3(d+.....)$, with no $(X-a)^2$ term, and $d \neq 0$.
Then the linear term giving the tangent line is just $c + b(X-a)$ as usual, but the $(X-a)^3$ term changes sign for $X<a$ as opposed to $X>a$. Thus the formula $f(X)$ goes from smaller than $c+b(X-a)$ to greater, as X passes through the value $X=a$. I.e. the line $Y = c+b(X-a)$ crosses the curve $Y = f(X)$ at $X=a$.

We could still define the tangent line algebraically to be given by the linear part of the equation, but we would also like a geometric description of the tangent line if possible. Fortunately Euclid also gave another description that works here as well.

In Proposition III.16, Euclid proves for a circle the familiar fact that a tangent line at P is perpendicular to the diameter ending at P, but he also proves something I had never realized. Namely he proves that if T is the tangent line to a circle at P, then no other line passing through P can be interposed between T and the arc of the circle at P.

I.e. Euclid shows that if T is the tangent line at P, and L is any other line through P, then L meets the circle at another point Q, and the arc of the circle between P and Q comes between the two lines L and T.

If you think about it, and apparently Newton did think about it a couple thousand years later, this says that the family of secant lines formed by joining P to other points Q of the circle, come closer and closer to T as the points Q come closer and closer to P. I.e. the tangent line T is approximated, "in the limit", by secant lines PQ, as Q approaches P along the curve. This is the modern definition of tangent line, a brilliant recasting of Euclid's Prop. III.16.

In our example above, the secant line given by the points P = (a,c), and Q = (X,f(X)), has slope given by

[f(X) - c]/(X-a) = [b(X-a) + (X-a)^3(d+.....)]/(X-a)

= b +(X-a)^2(d+....),

and this slope gets closer and closer to b, as (X-a) gets closer to zero, i.e. as X approaches a. I.e., the secant lines do get closer and closer to the line Y = c +b(X-a), so that is the tangent line.

So for that to happen, all that is needed is that the re-expansion gives us f(X) = c + b(X-a) + (X-a)^k.[d+.....] where k is any power greater than one.

I.e. with Euclid's old definition, we needed the power k to be even, for the line Y = c+b(X-a) to stay on one side of the curve, but for any power k > 1, that line lies closer to the curve than any other line through P = (a,c).

Thus taking the "linear part" of the curve near P does always give the tangent line in Newton's sense.

And since taking the linear part is additive, and since it also behaves well under constant multiples, we can short cut the tedious process of re expansion by doing it once and for all for the case of f(X) = X^n. I.e.

#30: Show that the linear part of X^n, re expanded in powers of (X-a), is n.a^(n-1).(X-a). Thus the slope of Y=X^n at (X,X^n), is nX^(n-1).

Then it follows by additivity and constant multiples, that the slope of
$Y = A+BX + CX^2 +DX^3 +...+EX^n$ at X, (the “derivative”)

is $B + 2CX + 3DX^2+....+nEX^{(n-1)}$.

With that we can compute the slope formula for any polynomial.
But what should we do to compute the slope of $Y = 1/X$? at $X=a$.

One possibility: use the geometric series:

I.e. if $S = 1+r +r^2 + r^3 +.....$ then $rS = r + r^2 + r^3 +r^4+....$
so then $S-rS = 1$, so $S(1-r) = 1$, so $S = 1/(1-r) = 1+r +r^2+r^3+.......$
This is called the geometric series, and it makes sense for $-1 < r < 1$.

Thus we can expand $1/X$ about $X=a$, as $1/[a -(a-X)] = (1/a)(1/[1 -(a-X)/a])$
$= (1/a)(1 +(a-X)/a + (a-X)^2/a^2 +......)$,
hence the linear part is $(1/a)(1 - (X-a)/a)$, with slope $-1/a^2$.
(The series makes sense for $-a < (X-a) < a$, i.e. $0 < X < 2a$.)

Another possibility: Do as Newton did: approximate the tangent line by secant lines and try to guess what their slopes are getting closer to, since that must be the slope of the tangent line.

For the curve $Y = 1/X$, with points $P = (a, 1/a)$, and $Q = (X, 1/X)$, (by the two point formula for slope) we get a secant with slope
equal to $[1/X - 1/a]/(X-a)$.

Now we have to let point Q get closer to point P and try to see where this slope is headed.

It helps to simplify the fraction by removing denominators, i.e. multiply by (aX/aX), to get $[a - X]/[aX(X-a)]$.

And now we can simplify more by canceling $(X-a)$ from top and bottom, getting $-1/aX$.

Now as Q gets closer to P, i.e. as X gets closer to a,
this seems to be getting closer to $-1/a^2$, exactly the same result as we obtained above from the geometric series.

Notice then the slope formula for $1/X = X^{(-1)}$, follows the same pattern as for positive powers,

i.e. the slope is $(-1)(X^{(-2)})$, which is $nX^{(n-1)}$ for $n = -1$.

#31: use one of the methods above to show the slope formula for $1/X^n = X^{(-n)}$, is as expected, $(-n)X^{(-n-1)} = -n/X^{(n+1)}$.

another typical calculus problem:
#32. A rectangular pea patch is to be constructed with area 216 sq.m., surrounded by a fence and also divided in half by another length of fence running through the middle and parallel to one of the outer sides. Find the dimensions of the outer rectangle that will minimize the total length of fencing needed, and find that minimal total length.

Suppose two parabolas are $y=x^2+8x+4$ and $y= -x^2+5x+2$.
#33. Show they do not meet, but that they do have some common tangent lines. I.e. there are some lines that are tangent (at different points) to both parabolas. Discover how many such lines are there, and find those lines.

I also need to clarify some things I said that are not quite precise.
When I ask whether a tangent line “crosses the graph”, I don't mean exactly that. I only mean to ask whether it “crosses the graph at the point of tangency”. I don't care whether it may or may not cross again elsewhere. So I am looking exactly for points at which the curve changes from concave up to concave down, or vice versa. Tangency is a local phenomenon, only relevant to how the line meets the curve very near the given point.

E.g. the graph of $Y = X^4 - X^2$ has tangent line $Y=0$ at $(0,0)$. This line does not cross the graph at that point, i.e. it stays above the graph very near that point and on both sides of that point, although that line does cross the graph away from the point of tangency, namely at $(1,0)$ and $(-1,0)$.

So a line that crosses a graph may do so either tangentially or transversely, and I am only asking about tangential crossings. They are a little unusual. A graph is concave up if the tangent line is getting steeper there, so the slope will be increasing, and it is concave down where the slope is decreasing. So where it changes from concave up to concave down, the slope will be neither increasing nor decreasing.

Thus the slope of the slope should be zero at such a point. So for X^4-X^2, the slope is 4X^3-2X, so the slope of the slope is 12X^2 - 2, which is zero only at ± sqrt(1/6). So there are at most these two points where the tangent line crosses the graph tangentially.

The tangent line at each of these points does cross the graph again transversely elsewhere.

Another way to look at this phenomenon, is another point of view on tangency. This is a nice algebraic aspect of tangency we may not have mentioned yet. I.e. a tangent line meets the curve possibly several times, so we can set the two equations equal, the one for the line and the one for the curve, to find those intersections.

When we solve those equations, some roots may be multiple roots, i.e. the corresponding factors may occur more than once. A line is tangent to a curve at a point precisely when that point occurs as a multiple intersection, i.e. at least twice.

So a tangent line is a line that meets a curve "more than once" at a single point. But it can also meet more than twice. A line that meets a curve exactly three times at the same point will cross the curve at that point.
In the case above Y = X^4 - X^2, meets Y=0 (the tangent line at (0,0)), exactly at the solutions of X^4 - X^2 = 0, namely at the 4 roots X = 0,0,1,-1. One of these intersection points counts doubly, the one at X=0, and that is where the line Y=0 is tangent to the curve Y = X^4-X^2.
It does not meet it triply there however and that says the line stays on one side of the curve near that point.

At the point X = sqrt(1/6), the tangent line does meet the graph triply.
An easier example is the curve Y = X^4 + X^3, which has tangent line Y=0 at (0,0), and that line meets the curve triply at X=0, since X=0 is a triple root of X^4+X^3 = 0. At that point the tangent line crosses the graph tangentially. Indeed it also crosses it once again (transversally) elsewhere, for a total of 4 intersections, 3 at (0,0) and one elsewhere. The slope of the slope formula is 12X^2 + 6X, which also has a zero at X=0, and the concavity changes there from down to up.

I did not realize there could not be a 4th degree (quartic) equation with only one tangent line that crosses the graph at its point of tangency, but there cannot be. The reasoning that makes it visible to me is to note that a quartic (with positive lead term) is concave up at both ends of the graph, i.e. for large positive and negative values of X.

Hence the concavity changes an even number of times. I.e. there are an even number of points where the tangent line crosses the graph tangentially. (Those are called points of inflection, and the lines are called inflectional tangents. In general an inflectional tangent is one that meets the curve more than doubly, either an odd or an even number of times, at a point, so it may not cross the curve. I.e. $Y=0$ is also an inflectional tangent to $Y = X^4$ at (0,0). Note that the curve $Y=X^4+X^2$ does not have an inflectional tangent at (0,0) even though the tangent line meets only there in the real plane, rather that tangent line meets it again at the imaginary points i ,-i.)

#34. Given parabolas $y=x^2+8x+4$ and $y=-x^2+5x+c$, find c so that the two parabolas intersect exactly once. Then find their tangent lines at that point.

Suppose the two parabolas are $y=x^2+8x+4$ and $y=-x^2+5x+2$.
then the tangent line to the first parabola at (a, a^2+8a+4) has equation
$Y - a^2+8a+4 = (2a+8)(X - a)$.

This line meets the second parabola at the common intersection points of that line with the parabola
$Y = -x^2+5x+2$.

Find a such that this intersection consists of only one point.

Here are a few more calculus type problems:
#35. Show that a rectangular box with closed top and square base and fixed total area A, has maximum volume if the height is equal to the base edge.

#36. If the box instead has an open top does it change anything?

#37. What are the dimensions of the right circular cylinder with largest volume, that fits inside a hemisphere of radius 10? (Assume the base of the cylinder lies in the plane of the equator, i.e. in the base of the hemisphere.)

We want to use slope formulas, i.e. "derivatives", to understand "Descartes' rule of signs", which we realize is actually due to somebody less famous.

First, we say that a polynomial $f(X)$ has a "root of multiplicity k" at $X=a$, if $(X-a)$ occurs as a factor of $f(X)$ of order precisely k. I.e. if $f(X) = (X-a)^k.g(X)$, where $g(a) \neq 0$.

#38: i) Show that a polynomial has a root of odd order at $X=a$ if and only if the graph of $Y=f(X)$ crosses from one side of the X - axis to the other as X goes from slightly less than a to slightly more than a. I.e. show that $f(X)$ changes sign as X passes through a root of odd order.

ii) Show $f(X)$ has a root of even order at $X=a$ if and only if the graph of $Y=f(X)$ stays on the same side of the X - axis as X passes through a.

#39. Explain why a polynomial of even degree, with real coefficients, has an even total number of real roots, provided we count each root with its correct multiplicity in the sense defined above. E.g. $f(X) = X^4$ has only one root, at $X=0$, but that root has multiplicity 4, so the total number of real roots, counted properly, is 4.

#40. Prove the following weak form of Descartes rule of signs: If the lead coefficient and constant term of a polynomial $f(X)$ have the same sign, then:
i) the coefficients of $f(X)$, taken in order (by degree) from lead coefficient to constant term, have an even total number of sign changes, and

ii) the polynomial $f(X)$ also has an even number of positive real roots, provided these are each counted with their correct multiplicity.
iii) Similarly, if the lead coefficient and constant term have opposite sign, then there are an odd number of sign changes and an odd number of positive real roots, counting each root properly.

For a polynomial $f(X)$, denote its slope formula, or "derivative" by $f'(X)$ or df/dX. E.g. if $f(X) = X^3$, then $f'(X) = df/dX = 3X^2$.

#41. i) If a polynomial $f(X)$ has roots at $X=a$ and $X=b$, explain why its derivative $f'(X)$ must have at least one root at some point c with $a < c < b$.

ii) Could f' have more than one root between a and b?

iii) If f has a root at $X=a$ of order 2 or more (i.e. if $a=b$), explain why f' also has a root at $X=a$.

Assuming the results of all previous problems, show why:

#42. If a polynomial has more positive roots than sign changes in its coefficients, then it has at least two more positive roots than sign changes.

#43. If a polynomial has more positive roots than sign changes then its derivative also has more positive roots than sign changes.

#44. If there is a polynomial with more positive roots than sign changes, then there is one of lowest degree with that property. Why does that lead to a contradiction?

Just to get it straight from the horses mouth, here is:

Newton's definition of tangent to a curve: cf. pp. 56-57 of his Principia, Book I.
"1) If a straight line meet a curve in 2 points A,B, and if B move up to A, and ultimately coincide with A, AB in its limiting position, is a tangent to the curve at point A.
2) The tangent is the direction of the side of the polygon, of which the curve is the curvilinear limit, when the number of sides are increased indefinitely.
3) The tangent to a curve at any point, is the direction of the curve at that point."

The following is a slight paraphrase from p.59:
If P is a point on a curve with coordinates (a,b) and Q is another point of the same curve with coordinates (X,Y), then the line joining P to Q has slope (Y-b)/(X-a), and hence the slope of the tangent line at P is the limiting value of that ratio, as (X,Y) approaches nearer to (a,b) than any fixed distance.
You can get the free google book at:
http://books.google.com/books/about/New … g2AAAAMAAJ

Challenge: try to generalize the problem on maximizing the volume of a closed top rectangular box of total area A, to the case where we do not assume the base is square.
I.e. assume the dimensions are L,W,H, with the base being L by W, with height H. then assume L = cW, where c is a constant ratio.
Show the maximum volume that occurs is (A/6)^(3/2).(2sqrt(c)/c+1).
Since in the square base case the maximum volume was (A/6)^(3/2), that will always be the absolute maximum, provided we can show that for all 0 < c ≠ 1, we have 4c < (c+1)^2, i.e. 2sqrt(c) < (c+1).
show that.

Here is a quick review and summary of tangent lines.
Intuitively, Euclid said a tangent line (to a circle) is a line that meets the circle but does not cross it.
Euclid's definition works for all curves of degree two, like circles, ellipses and parabolas but not perfectly for higher degree curves. If we understand "does not cross it" to mean only locally near the point, but we allow possible crossing further away, then this definition works again for most points on most curves.

But most higher degree curves have a few points where the tangent line does cross the curve right at the point of tangency, like the line $y=0$ for the curve $Y=X^3$, at the point $(0,0)$.
So we have to look more closely to define tangency in general. We gave three points of view:

i) to find the tangent at $X=a$, re expand the equation for the curve in powers of $(X-a)$ instead of X, and pick off the linear terms. this works for all polynomials.

ii) Define intersection multiplicity of a line and curve, at a point $X=a$, by substituting the equation for the line into that of the curve and considering how many factors of $(X-a)$ one gets, i.e. how many roots $X=a$ should count as. This also works for all polynomials.

iii) Use Newton's sophisticated version of one of Euclid's results (Euclid's Prop. III.16) and define a tangent line at $X=a$ as a limit of secant lines, obtained by choosing a second value of X, call it just X, and taking the tangent to be the limiting position of the secants as the second point $(X,f(X))$ comes closer and closer and ultimately coincides with $(a,f(a))$.

All three of these approaches lead to the same formulas for slopes; e.g. the curve $Y = X^n$ has slope $nX^{(n-1)}$ at the point (X,X^n). The curve $Y = X^{(-n)}$ has slope $(-n)X^{(-n-1)}$ at the point $(X,X^{(-n)})$.

And the slope formulas add naturally. I.e. the slope formula for $f+g$ is the sum of the slope formulas for f and for g. Moreover if c is a constant, the slope formula for $cf(x)$ is c times the slope formula for $f(X)$.

This allows us to write slope formulas for all polynomials at least. We write f' for the slope formula of f.

Here is something new, the slope formula for a product. If we have $f(X) = a + bX + cX^2 + \ldots$, and $g(X) = m + nX + pX^2 + \ldots\ldots$, then what is the slope formula for fg? (the product of f and g.)

It is easy at $X=0$, since all we want is the linear term. I.e. then $fg = am + (an+mb)X + (ap+bn+cm)X^2 + \ldots$.

so the linear term is $am + (an+mb)X$, and the slope at $X=0$ is $an+mb$.

Notice this is the value $f(0) = a$, of f at $X=0$, multiplied by the slope $g'(0) = n$ of g at $X=0$, plus the same thing the other way around, i.e. plus the value $g(0) = m$, times the slope $f'(0) = b$.

The same pattern should hold for any re expansion, i.e. the linear term of a product is always the sum of two pieces, each one being the product of a constant term and a linear term.

This gives us the "product rule" for slope formulas: $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Notice this works for the product $X^2 = XX$, since then at $X=a$ we get $a.1 + 1.a = 2a$, or at X, we get just $1.X + X.1 = 2X$.

Here is a reference for a version of Descartes' method of finding tangents from my own class notes:
http://www.math.uga.edu/~roy/polynomial … limits.pdf

Now we have enough tools to return to multiplicities of roots and the rule of signs.

For #40, A good question to start off with, which I really should have asked first, is this:
Why must a polynomial of odd degree always cross the X axis at least once?

Here's another thing to think about that is related: it seems that a polynomial of even degree, say, with 4 positive roots (and 4 sign changes), can be made to have no real roots at all, just by changing the constant term. Does this seem true?

I.e. you can reduce the number of positive real roots to zero, but only take away at most one sign change. Look at some examples and see if you believe this.

Can you say also that an odd degree polynomial has at least one root of odd multiplicity? pourquoi?

I used to hear that a polynomial has a (real) root at a point where its graph crosses the X axis. That is true of course, but it is also true that a root occurs at a point where the graph touches the X axis without crossing it. We have learned that these two types of roots correspond to roots of odd multiplicity and roots of even multiplicity.

So to understand whether the total number of real roots a polynomial has is even or odd, we need to know how many times its graph crosses as opposed to touching without crossing.

#42,43,44 seem a bit hard and involve several steps at once. If desired, put them on hold briefly and take some related problems that may lead your thinking in the right direction for those.

#45. i) Show that the graph of a polynomial of odd degree must cross (from one side to the other) the X axis an odd number of times (first show it must cross at least once).

ii) Show that the graph of a polynomial of even degree must cross the X axis, if it does so at all, an even number of times, (possibly zero).

Recall that real roots have either odd multiplicity or even multiplicity.
#46. i) Show that if a polynomial has odd degree, then the number of its roots having odd multiplicity is also odd.
ii) Show that if a polynomial has even degree, then it has an even number (possibly zero) of real roots of odd multiplicity.

#47. i) Assuming the previous results, show that if a polynomial has odd degree, then the total number of its real roots is odd, provided each root is counted with its multiplicity, (e.g. we count a triple root as 3 roots, and a double root as two roots, etc...).

ii) Show that a polynomial of even degree has an even total number of real roots, when each root is counted with its multiplicity.

A hint for #42:

If the lead term and constant term of a polynomial have the same sign, (no matter what the degree), explain why the graph crosses the X axis an even number of times, on the right side of the Y axis, i.e. for positive X.
Then explain why the total number of positive roots, each counted with its multiplicity, is even.

(Since we already know the number of sign changes in the coefficients is also even, this says the difference between the number of coefficient sign changes and the number of positive roots, is the difference of two even numbers.)

An algebraic argument that complex roots always come in pairs: suppose that z is a complex root of a polynomial such as $aX^3 + bX^2 + cX + d$, where a,b,c,d are real numbers.

I.e. suppose that $az^3 + bz^2 + cz + d = 0$. Then the "complex conjugate" of both sides of this equation must be equal too, (where the complex conjugate of $u+iv$ is $u-iv$).

The complex conjugate of zero is zero, and the complex conjugate of a product or sum is the product or sum of the conjugates, so if w^* is the conjugate of w, then the conjugate of the left side of the equation is:
$a^*(z^*)^3 + b^*(z^*)^2 + c^*(z^*) + d^*$.

Thus $a^*(z^*)^3 + b^*(z^*)^2 + c^*(z^*) + d^* = 0^* = 0$.

But the conjugate of a real number is the same real number. So $a^* = a$, $b^*=b$, $c^*=c$ and $d^*=d$.

Thus we have $a(z^*)^3 + b(z^*)^2 + c(z^*) + d = 0$, i.e. z^* is also a root of the same equation that z was a root of. So indeed, the conjugate of a root of an equation is also a root, provided the coefficients are all real.

Of course one can see they come in pairs also by using the quadratic formula on quadratic factors. But some argument should then be given that every real polynomial does factor into linear and quadratic factors.

So in these problems, I am sort of pretending we don't know about complex numbers, and trying to get the same results using multiplicities of real roots. I.e. if we show that the total number of real roots, with multiplicities, of an odd or even degree polynomial is respectively odd or even, then once we know the total number of all roots, real or complex, is the same as the degree, it would follow that the total number of non real roots is even.

It is reasonable to wait a bit before assuming the fundamental theorem that every polynomial has a complex root, since that is harder to prove than what we are doing. It also uses similar ideas but subtler. I.e. we are using ideas about the number of times a curve crossing an axis in the plane to get to the other side, while the related proof of the fundamental theorem uses the idea of the number of times a closed curve winds around a point.

I.e. the following principle has been crucial to answering most problems: “the IVT (intermediate value theorem)”, which says that a continuous function f, defined for $a \leq x \leq b$, and such that $f(a) < 0$ and $f(b) > 0$, must also have a point c with $a < c < b$, such that $f(c) = 0$.

Geometrically, this just says that a continuous graph that starts out below the X axis and ends up above the X axis, has to cross the X axis somewhere in between.

A basic fact is that all polynomials f(X) are continuous. This means intuitively that if X only changes a little then f(X) also changes only a little. A precise way to say f is continuous at a point c, is that if $f(c) > 0$, then $f(X) > 0$ for all X close enough to c.

For us the consequence we care about is the intermediate value property. So we are using the fact that a polynomial that assumes the values u and v, also assumes all values between u and v.

The insight that an odd degree polynomial has tails pointing in opposite directions at both ends, coupled with the fact that those tails go up and down without any bound above or below, implies that an odd degree polynomial has a graph that will be above the X axis at some point, and below the X axis at some other point.

Then the IVT implies that it must cross the X axis somewhere.
As we know from some other problems above, this not only means it has a "root" there, but that is has a root of odd multiplicity.

Do you see how it is possible from this perspective, for even degree polynomials to have no roots at all? and why an even degree polynomial with positive lead coefficient always has a minimum value? (I.e. its graph has a lowest point).

The analysis of coefficient sign changes and positive roots is very interesting. In particular, for linear functions, I can see why it must be true.

With higher degree polynomials, I am not able to “see” for sure why another root must be accompanied by another sign change.

Here is the kind of reasoning that I have seen used for this argument. It is an "inductive" argument, proceeding upwards "one degree at a time".

I.e. assuming you have proved it for linear polynomials, try to derive it for quadratic ones.

Suppose f(X) = aX^2 + bX + c, where both a and c are positive. Then by the kind of argument we have been giving, the graph crosses the X axis an even number of times, and hence has a even number of total roots, counted with multiplicities.

But in this case, also there are an even number of coefficient sign changes. I.e. if b >0 there are none, and if b < 0 there are two.

Now suppose there were a quadratic polynomial like this that violated the "rule of signs". I.e. it would have more positive roots than sign changes. Since both those numbers are even, it would have at least two more roots than sign changes. E.g. if there are no sign changes,
if it has some positive roots, it would have to have two of them.

But we also know the derivative f' has a root in between those two roots of f.
And f' = 2aX + b, also has no sign changes.
This is a contradiction to the case of linear functions, which we assume has already been proved.

We also need to cover the case where a and c have opposite signs, but it is similar.

Thus, using all our detailed knowledge of evenness and oddness, both for degrees and multiplicities, we are showing that: if there were a quadratic polynomial that violates the rule, there would also be a linear one that violated it.

This argument can be used to give a proof by "infinite descent", similar to Euclid's proof that sqrt(2) is irrational.

Let's proceed next with some problems on area.

#48. Probably everybody has heard the area of a triangle is (1/2)BH, where B,H are the lengths of the base and height. Assuming we agree that the area of a parallelogram of base B and height H is BH, explain why the formula for a triangle is correct. Hint: The formula is equivalent to showing that the area of two copies of the triangle is BH.

#49. Suppose we want to know the sum of the first n integers, 1+2+3+....+n = S1.
Imagine this as the area of a triangular array of blocks, with one block in the first row,
two blocks in the second row, three in the third row,..., and n blocks in the nth row. Find a formula for the sum of these integers. Hint: This question is related to the previous one.

#50. Here is another cool way to find the sum of the integers 1+2+3+....+n = S(1).
Look at the formula (k+1)^2 - k^2 = 2k + 1. Then substitute n for k, to get (n+1)^2 - n^2 = 2n+1.

Next substitute n-1 for k, to get
(n-1+1)^2 - (n-1)^2 = n^2 -(n-1)^2 = 2(n-1) + 1.

Next substitute n-2 for k to get
(n-2+1)^2 = (n-1)^2 - (n-2)^2 = 2(n-2) + 1.

Keep going until we reach
2^2 - 1^2 = 2(1) + 1.

If we add up the left hand side of all these equations. What do we get?

Adding up the right hand sides of these equations, show we get 2.S(1) + n.

How does this give a formula for S(1)? Is it the same as the formula we got in #49?

#51. The previous method suggests a generalization to the case of the sum of squares. We can find a formula for the sum

1^2 + 2^2 + 3^2 + ...+ n^2 = S(2), as follows:

Check that
$(k+1)^3 - k^3 = 3k^2 + 3k + 1$. Now substitute n, $n-1$, $n-2$, etc.., down to 1, for k, for k, getting:

$(n+1)^3 - n^3 = 3n^2 + 3n + 1$,

$n^3 - (n-1)^3 = 3(n-1)^2 + 3(n-1) + 1$,

$(n-1)^3 - (n-2)^3 = 3(n-2)^2 + 3(n-2) + 1$,
....
$3^3 - 2^3 = 3(2)^2 + 3(2) + 1$,

$2^3 - 1^3 = 3(1)^2 + 3(1) + 1$.

Adding up the left hand sides gives ??
Show that adding up the right hand sides gives
$3.S(2) + 3.S(1) + n$.

Use this to show that $S(2) = (1/6)(n)(n+1)(2n+1)$.

#52. Now, compare the formulas you derived above, namely:
$(n+1)^2 - 1^2 = 2S1 + n$, and

$(n+1)^3 - 1^3 = 3S2 + 3S1 + n$,

with the coefficients in the binomial formula: $(a+1)^2 - a^2 = 2a + 1$, and $(a+1)^3 - a^3 = 3a^2 + 3a + 1$, to try to guess what you would get if you did the same calculation for:

$(n+1)^4 - 1^4 = ????$

Hint: look at the binomial formula for $(a+1)^4$. Using this deduce a formula for S3, without going through all the work above,
and then check it against some specific cases, like $1^3 + 2^3$, $1^3 + 2^3 + 3^3$,.......

#53. Do you notice a relation between S3 and S1? (this is apparently kind of a one time thing that nobody understands the reason for. if you see a way to generalize this, let me know, as that could be big news.)

Thinking back to problems #48-#50, there was a connection between the area of a triangle and the sum of the integers 1+2+3+...+n. In fact we can see how to find the area of a triangle using that summation formula S(1). Although it will be more complicated than the usual method in #48, it will be more general.

Start from the right triangle with base length 1 and height 1, with vertices at (0,0), (1,0) and (1,1), in the Cartesian plane, and approximate it by a triangular array of blocks as we used above to compute the sum of the integers 1+2+....+n.

So let the bottom row have n blocks, each of base length 1/n, and height 1/n. So each block has area 1/n^2, and since there are n of them, the area of this bottom row of blocks is n.(1/n^2). and I think we don't want to simplify this yet, no matter how tempting.

Notice the blocks extend beyond the triangle at the left base vertex.
Next add another row of (n-1) blocks of the same size on top of this row, starting at the x coordinate 1/n and continuing all the way to x=1. This row has total area (n-1)(1/n^2).

Continuing, we get a triangular pyramid of blocks that completely cover the original triangle. All those blocks have area 1/.n^2.
#54. Show there are S1 blocks all together, and the total area of the blocks is S1/n^2
= (1/2)(n)(n+1)/n^2 = (1/2)(n^2 + n)/n^2 = (1/2)(1 + 1/n)

#55. Explain why the area of these blocks is greater than that of the original triangle, no matter what n is, and conclude that the area of the triangle is at most equal to the "limiting value" that is being approached by the areas of these pyramids, as n grows larger and larger.

In particular, what is the limiting number that is being approached by the numbers (1/2)(1+1/n) as n grows and grows beyond all bound?
Is that the expected area of the triangle?

We have shown by approximations via rectangles (actually squares), that 1/2 is an upper bound for the area of a right triangle of base 1 and height 1. Big deal.

BUT: The method we used also shows this is true for any triangle that has every horizontal slice of the same length as such a triangle, namely any triangle of base one and height one, whether right or not. Do you see this?

I.e. our approximation method only,used the length of the slices at various heights, not how they were positioned with respect to each other horizontally.   This is the basic fact about area: any two figures on the same base, such that every horizontal slice taken at the same height, has the same length, also have the same area. This is the phenomenon we illustrate by sliding a deck of cards over at an angle, and claiming the deck still has the same volume.

I think I understand it now better than before. I.e. since a pyramid of blocks has the same area no matter how the various rows at different heights are positioned horizontally, so also does the triangle they approximate.

This principle, which was known to Archimedes, is usually called the "Cavalieri" principle, after a much later Italian mathematician. This is a common phenomenon in mathematics: ideas that were known much earlier by pioneers, are named after people who rediscovered them much later, but at a time when the majority of scientists could understand them.

#56. To get a lower bound for the area of our triangle, take a row of blocks of base length 1/n, but beginning at X = 1/n instead of at X=0. Then we only get n-1 blocks in the bottom row, and the row lies entirely inside the triangle. Then continue on up, using n-2 blocks in the next row, etc.... until we stop one level below the top, with one triangle at height (n-1)/n. Then we have (1+2+3+....+(n-1)) blocks, each of area 1/n^2.

i) Show that the total area of this pyramid, which lies entirely inside the triangle, has area equal to (1/2)(n^2-n)/n^2 = (1/2)(1-1/n). What number does this approach as we take n larger and larger?

ii) Conclude that the area A of our triangle, satisfies 1/2 ≤ A ≤ 1/2, hence equals 1/2.

Whew!! what a lot of work to learn that! But we have made great strides as we shall see next. I.e. Does this suggest a possible connection between the sum of the squares 1^2 + 2^2 + 3^2+...+n^2 and the area under a parabola? How would you proceed?

#57.
I don't see how to generalize our exact method of computing area of a triangle to a parabola, but we can do it if we tweak the method a little. Let's try to simplify our method for the triangle a bit by using fewer blocks in our approximations.

Again look at the graph of the line joining (0,0) to (1,1), and this time approximate it by just rectangles. I.e. it is wasteful to stack all those squares on top of each other, since we could just use the vertical rectangle they form.

So the first rectangle has base the segment of the x axis from (0,0) to (1/n, 0), and height equal to 1/n. So this one is the same as the square we had there before.

But the next rectangle will be twice as high as before. I.e. it again has as base the segment from (1/n,0) to (2/n,0), but now has height 2/n. So this rectangle is made up of two of the previous squares stacked on top of each other.

Keep on like this. We get n rectangles, with areas:
(1/n)(1/n), (1/n)(2/n), (1/n)(3/n),.....,(1/n)(n/n), i.e. with areas:

(1/n^2), (2/n^2), (3/n^2),......,(n/n^2), and again we resist the urge to simplify.

Now add these up, use the formula for S1, and see what you get. Then simplify, let n-->infinity, and try to see what the limit is, (hopefully it = 1/2).

#58. Now we try to do the same rectangle approximation method for the area under the parabola Y= X^2, between X=0 and X=1. Presumably, the solution should involve the formula S2.

So the first rectangle has the same base as before, the segment from (0,0) to (1/n,0) on the x axis, and it reaches up to the parabola at5 the right end, so has height (1/n)^2. This rectangle has area (1/n)(1/n^2), and again we do not simplify.

The next rectangle has the same base as before, the segment from (1/n,0) to (2/n,0), and reaches up to the parabola over the right endpoint. Thus it has height (2/n)^2, and area (1/n)(2/n)^2.

Continuing, and adding up the areas, we seem to get:
(1/n)(1/n)^2 + (1/n)(2/n)^2 + (1/n)(3/n)^2 +....+(1/n)(n/n)^2.

Simplify this to show it equals, let me see now, maybe (1/n^3).S3 ??

Then use your formula for S3 to take the limit as n-->infinity and find out what the area under the parabola is. Before doing that, do you have a guess as to what it might be?

#59. Knowing the degree of a polynomial can be very useful, especially in taking limits.
Show that if f(X) is a polynomial of degree 2 or less, then
f(n)/n^3 -->0 as n-->infinity.
If f(X) is a polynomial of degree less than k, explain why
f(n)/n^k-->0 as n-->infinity.

#60. By the previous problem, we only need to know the lead term of the formulas Sk to use them find areas.
We worked hard to compute S3, but we could have computed just the lead term more easily.
I.e. We started from (n+1)^4 - 1^4 = 4S3 + 6S2 + 4S1 + n.

i) We know S2 has degree 3, and S1 has degree 2, so looking at this equation, explain why we get n^4 = 4S3 + (terms of degree 3 or less).
Conclude the lead term of S3 is n^4/4. Does this agree with what we found before?

#61. Use just the fact that S3 = n^4/4 + terms of degree 3 or less, to find the area under Y=X^3 from X=0 to X=1? First make a guess as to what the area is.

#62. What do you guess is the area under the parabola Y = X^2, from X=0 to X=2? Can you compute it by the rectangle method?

#63. i) What do you think is the area under Y=X^2 from X=0 to X=X, i.e. over the interval [0,X]? Can you compute it by the rectangle method?
iii) What is the derivative of this area formula?

#64
i) What do you think is the area under Y=X^n over the interval [0,X]?
Without computing the area formula, what do you think its derivative should be?

iii) Does the information in part ii) allow you to guess the area formula?

A digression on sequences:
I claim that if I know a sequence begins 1,2,3,5,7, and the nth term is given by P(n) where P(X) is a polynomial of degree 4 or less, then I can guess the polynomial. Namely:

P(X) = (X-2)(X-3)(X-4)(X-5)/(1-2)(1-3)(1-4)(1-5)
+2 (X-1)(X-3)(X-4)(X-5)/(2-1)(2-3)(2-4)(2-5)
+3 (X-1)(X-2)(X-4)(X-5)/(3-1)(3-2)(3-4)(3-5)
+ 5 (X-1)(X-2)(X-3)(X-5)/(4-1)(4-2)(4-3)(4-5)
+ 7 (X-1)(X-2)(X-3)(X-4)/(5-1)(5-2)(5-3)(5-4).

P(X) has 5 terms and each term is set up as follows: When you plug in X=1, all the terms are zero except the first, which has the same numbers in top and bottom of the fraction when X=1, so P(1) equals 1.

When you plug in X=2, all the terms are zero except the second term, which again would equal 1 except it has been multiplied by 2, so it equals 2.
Each time we plug in a number from X=1,..., to X=5, there is only one non zero term, which equals 1 but multiplied by the values we want, namely 1,2,3,5,7, respectively.

Amazingly, the formula also gives P(6)= 6, when I did it by hand, but I could of course be wrong. So I think P(X) gives 1,2,3,5,7,6, when X=1,2,3,4,5,6. ???

In particular these are all integers. I didn't simplify it, but I wonder if I did, would P(X) have integer coefficients?

By the way here is another cool polynomial noticed by Euler, that seems to give prime values for every X=1,...,. up to X=40.
f(n) = n^2 - n + 41. But when X=41, it gives ????

If you are curious to see the original explanation of the definition of definite integrals ("bestimmten integralen"), i.e. areas by approximating rectangles, by Riemann himself, even if you don't know German, I recommend looking at this link to his paper, specifically the bottom of page 12 and top of page 13 (section 4) where you recognize a Riemann sum.
http://www.maths.tcd.ie/pub/HistMath/Pe … g/Trig.pdf

In the next section 5, pages 13-15 he gives a condition that is necessary and sufficient for the limit of Riemann sums to exist. Apparently most people did not read this, and that same condition is mostly known today as "Lebesgue's criterion", after a French mathematician working some 50 years later on a more sophisticated theory of integrals.

Hint for the limits in the problems above: $S2 = (1/3)n^3$ + a quadratic term in n (like an^2+bn+c), so the limit can be taken separately as the limit of $(1/3)n^3/n^3$ + limit of $(an^2+bn+c)/n^3$.

After doing the previous limit, try to write down the analogous “Riemann sum” for the area from X=0 to X=X, (as opposed to X=0 to X=1?)
I.e. your Riemann sum will have X in it.

Hint: Divide up the interval [0,X] into subintervals [0,X/n], [X/n,2X/n], [3X/n],.....,[nX/n].

Then what will the height of the graph be over the point kX/n , (where $1 \leq k \leq n$?)
How does the result differ from the previous case of [0,1]?

(This part is key to the fundamental theorem of calculus.)

Now the moral of this story is that the derivative of the area formula for X^2 is X^2 again.

This is true for all powers of X, i.e. one can see using the same arguments as above that the lead term of S(k) is $n^{(k+1)}/(k+1)$. Does that look right? and the limit only depends on that term, since all the others go to zero when divided by $n^{(k+1)}$.

And thus the limit of the Riemann sums for $Y = X^k$, between X=0 and X=X, should be $X^{(k+1)}/(k+1)$. So again the area function is an antiderivative of the height function.

Hence this holds all polynomials, since (one can see that area is additive. What do you think? We can thus see the “fundamental theorem of calculus” directly in these cases.

I.e. you may know the general principle, that the derivative of the moving area function, is the height function.
Have you heard of that? I.e. the area function is an antiderivative of the height function.
This is called the (first) fundamental theorem of calculus. So we are proving the first fundamental theorem of calculus for polynomials.

Another way to look at it is to say the derivative of the moving "area under the graph" function, is equal the length of the slice of that area, taken perpendicular to the X axis.

65. If that makes sense, here is another analogous question: Suppose we consider the unit radius disc, centered at the origin, and revolve the part of it lying in the upper right quadrant of the x,y plane, around the X axis, to generate a hemispherical ball in 3 space.

Now consider the "moving volume function" for this ball, as a function of X. I.e. V(X) is the volume of that part of the ball lying between X=0 and X=X. What do you think the derivative of that moving volume function might be? i.e. V'(X) = ??

Hint: in the case of the moving area function for a parabola, the derivative is the size (length) of the "leading edge" of the 2 dimensional region we are taking the area of, between 0 and X.

In the case of the moving volume function for a half ball, what is the "leading face" of the 3 dimensional region that we are taking the volume of, between 0 and X? what is its size?

Or to get the derivative of V at X, take a slice of the ball, perpendicular to the X axis. What is the shape, and size, of that slice?

To put it another way, for areas, the derivative of the area function was the size (length) of the leading edge of the moving region whose area was being measured, i.e. the length of the part of that region having X coordinate exactly equal to X.

So for volume presumably V'(X) has something to do with the geometry of the part of the solid having X coordinate equal to X also.

End of problems for (pre) epsilon camp 2013.