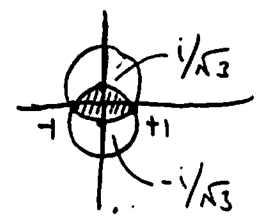
COMPLEX ANALYSIS QUALIFYING EXAM, SPRING 2013
JANUARY 4, 2013

1. a) Let u be a real-valued harmonic function on a simply connected domain $D \subset \mathbb{C}$. Prove that there is a function v (the *harmonic conjugate* of u) defined on D such that $f = u + iv$ is holomorphic.

b) Find the harmonic conjugate of $u(x+iy) = xe^x\cos(y) - ye^x\sin(y)$.

2. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$.

3. Let D denote the intersection of the two open disks $|z \pm i/\sqrt{3}| < 2/\sqrt{3}$. Find a one-to-one conformal map from D onto the half-plane $Re(z) > 0$.



Problem 3.

4. a) Prove that on the complement (in $\mathbb{C}$) of the interval $[-1,1]$, there is a holomorphic function $f(z)$ such that $e^{f(z)} = \dfrac{z-1}{z+1}$. (Hint: Consider the image of the right hand side. Alternatively, take the derivative of both sides.)

b) Show furthermore that f cannot be extended holomorphically to the complement of $\{-1,1\}$.

c) Find the Laurent expansion of f in the annulus $1 < |z| < \infty$. Note that f is not unique – your answer will have a parameter.

5. Prove that for every nonnegative integer n, the polynomial $f_n(z) = \displaystyle\sum_{k=0}^{n} \frac{z^k}{k!}$ has no roots in the open unit disk. (Hint: Check $n = 1$ and $n = 2$ directly.)

6. Consider the function $f(z) = \dfrac{1}{\sin z}$, defined on $\mathbb{C} - \pi\mathbb{Z}$.

a) Show that f cannot be approximated uniformly by polynomials on compact subsets of $\mathbb{C} \setminus \pi\mathbb{Z}$, i.e. there is no sequence $p_1(z), p_2(z), \ldots$ of polynomials such that for every compact $K \subset \mathbb{C} \setminus \pi\mathbb{Z}$ the sequence $p_n \to f$ uniformly on K.

b) Show that, on the other hand, it *is* possible to approximate f uniformly by polynomials on any compact disk contained in $\mathbb{C} \setminus \pi\mathbb{Z}$.