## ALGEBRA QUALIFYING EXAM, FALL 2022

**Directions:** Each part of each problem is worth $\frac{12}{n}$ points, where $n \in \{2, 3, 4\}$ is the number of parts of the problem. You may use the result of any part of a problem in your solution to any subsequent part, whether you solved the previous part correctly or not.

**1.** Suppose $G$ is a finite group acting on a set $X$.

(a) Let $x \in X$. Define the $G$-orbit of $x$, $\mathrm{Orb}_G(x)$, and the $G$-stabilizer of $x$, $\mathrm{Stab}_G(x)$.

(b) Prove that every orbit is a finite set and its cardinality divides the order $|G|$ of the group.

(c) Let $g \in G$. Prove that the set of distinct conjugates of $g$ has cardinality that divides the order $|G|$ of the group.

(d) Prove Cauchy's Theorem: Suppose $G$ is finite and $p \mid |G|$ for some prime $p \in \mathbb{Z}$. Then $G$ has an element of order $p$.
(Suggested outline: Define and study an action of the cyclic group of order $p$ on the set $X = \{(g_1, g_2, \ldots, g_p \in G^p \mid g_1 g_2 \cdots g_p = 1\} \setminus \{(1, 1, \ldots, 1)\}$.)

**2.** Let $m \in \mathbb{R} \setminus \{0\}$, let $\tau_1$ be the isometry of $\mathbb{R}^2$ given by reflection through the line $y = 0$, let $\tau_2$ be the isometry of $\mathbb{R}^2$ given by reflection through the line $y = mx$, and let $G \coloneqq \langle \tau_1, \tau_2 \rangle$ by the subgroup of isometries of $\mathbb{R}^2$ generated by $\tau_1$ and $\tau_2$.

(a) Find necessary and sufficient conditions on $m$ for $G$ to be finite.

(b) When $G$ is finite, show that it is isomorphic to a dihedral group $D_n$ of order $2n$ (and explain what $n$ is in terms of $m$).

**3.** Let $R$ be a commutative ring with 1.

(a) Let $S$ be a subset of $R$ that contains 1 and that is closed under multiplication. Show that if $I$ is an ideal of $R$ that is maximal with respect to the exclusion of $S$ (i.e., such that $I \cap S = \varnothing$), then $I$ is a prime ideal.

(b) Let $S$ be as above, and suppose moreover that for all $x, y, z \in R$, if $x = zy$ and $x \in S$ then also $y \in S$. Show that $R \setminus S$ is a union of prime ideals.

**4.** If $F$ is a field, $V$ is an $F$-vector space, and $T : V \to V$ is an $F$-linear endomorphism, then a **T-invariant subspace** is an $F$-subspace $\{0\} \subsetneq W \subsetneq V$ such that $T(W) \subseteq W$.

(a) Let $R$ be a commutative ring with 1, and let $M$ be a simple $R$-module. Show: there is a maximal ideal $\mathfrak{m}$ of $R$ such that $M \cong R/\mathfrak{m}$.

(b) Let $F$ be a field of characteristic 0, and let $n \geq 2$ be an integer. Show that the following are equivalent:
  (i) There is a linear map $T : F^n \to F^n$ that has no $T$-invariant subspace.
  (ii) There is a field extension $K/F$ of degree $n$.

(c) Show that if $n \geq 3$, every linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ has a $T$-invariant subspace.

**5.** Let $n \in \mathbb{Z}^+$, and put $\zeta_n := e^{2\pi i/n}$.

(a) Show that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is a Galois extension.

(b) Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. Show that $\sigma(\zeta_n) = \zeta_n^a$ for some integer $a$ that is coprime to $n$. Deduce that there is an injective group homomorphism

$$\mathrm{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \hookrightarrow (\mathbb{Z}/n\mathbb{Z})^\times.$$

(c) Show: $2^{1/3} \notin \mathbb{Q}(\zeta_n)$.

**6.** Let

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 2 & -1 \\ 2 & 0 & 0 \end{bmatrix} \in M_3(\mathbb{C}).$$

(a) Find the Jordan canonical form $J$ of $A$.

(b) Find an invertible matrix $P$ such that $P^{-1}AP = J$. (You should not need to compute $P^{-1}$.)

(c) What is the minimal polynomial of $A$?