Topology Qualification Exam, Fall 2021

Instructions:

(a) Please work on **8 out of 9** problems, and clearly mark which one you do not want us to grade.

(b) You can assume homology groups and fundamental groups of a *point* and *wedges of spheres in all dimensions.* Everything else should be computed.

1. Prove or disprove the following:

   (a) If X and Y are path-connected, then $X \times Y$ is path-connected.

   (b) If $A \subset X$ is path-connected, then its closure $\overline{A}$ is path-connected.

2. Let S be a connected metric space with metric d. Given $p \in S$, show that if $S \setminus \{p\} \neq \emptyset$ then $S \setminus \{p\}$ is not compact.

3. Given an example of a continuous map $f : X \to Y$ between connected spaces that is a continuous bijection but not a homeomorphism.

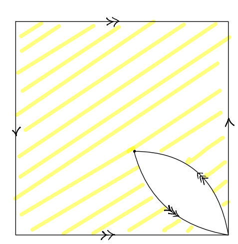
4. (a) Compute fundamental groups of T^3 and $\mathbb{R}P^3$ (Hint: construct their universal covers.).
   (b) Prove there is no covering map from T^3 to $\mathbb{R}P^3$.

5. Let Σ_g denote the surface of genus g.

   (a) Suppose there is a degree n covering map $f : \Sigma_g \to \Sigma_h$. What is the relationship between g, h and n?

   (b) Show that there is no finite covering map from Σ_{g+1} to Σ_g for $g > 2$.

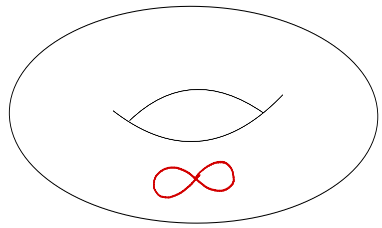
6. Let X be the topological space obtained from the Klein bottle K by removing a small open disk and identifying antipodal points of the resulting boundary circle on K as in the following figure.



   (a) Use Van Kampen's theorem to find a presentation for $\pi_1(X)$.

   (b) Compute the homology groups using cellular homology.

7. Let X be the topological space obtained by gluing the boundary of a disk to a torus along a figure eight shape curve as in the following figure. Use the Mayer-Vietoris sequence to compute the homology groups of X.



8. (a) Compute the homology groups of $X = S^2 \times S^4$ and $Y = \mathbb{C}P^2 \vee S^6$.

   (b) Show that X and Y are not homeomorphic.

9. Consider the torus T in $\mathbb{R}^3$ obtained by revolving the circle $(y-2)^2 + z^2 = 1$ in the yz-plane around the z-axis. Let i be the map induced by 180°-rotation around the y-axis on this torus i.e.,

$$i(x, y, z) = (-x, y, -z)$$

   (a) Find a cell structure on T such that i maps cells to cells.

   (b) The quotient of T with the relation $x \sim i(x)$ for all $x \in T$ is an orientable surface (you do not need to show this, you can take this as given). Find the genus of this surface.