Topology Qualifying Examination
Fall 2014

Justify all the calculations and state the theorems you use in your answers. The problems have equal weight.

1. (a) Define what it means for a topological space to be:
(i) connected
(ii) locally connected

(b) Give, with proof, an example of a space that is connected but not locally connected.

2. Is every product (finite or infinite) of Hausdorff spaces Hausdorff? If yes, prove it. If no, give a counterexample.

3. Let X and Y be topological spaces and let $f : X \to Y$ be a function. Suppose that $X = A \cup B$ where A and B are closed subsets, and that the restrictions $f|_A$ and $f|_B$ are continuous (where A and B have the subspace topology). Prove that f is continuous.

4. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

5. Prove that every continuous map $f : \mathbb{R}P^2 \to S^1$ is homotopic to a constant. (Hint: think about covering spaces.)

6. Compute the integral homology groups of the space $X = Y \cup Z \subset \mathbb{R}^3$ which is the union of the sphere $Y = \{x^2 + y^2 + z^2 = 1\}$ and the ellipsoid $Z = \left\{x^2 + y^2 + \frac{z^2}{4} = 1\right\}$.

7. Identify (with proof, but of course you can appeal to the classification of surfaces) all of the compact surfaces without boundary that have a cell decomposition having exactly one 0-cell and exactly two 1-cells (with no restriction on the number of cells of dimension larger than 1).

8. Prove that, for every continuous map $f : B^2 \to B^2$ there is a point x such that $f(x) = x$.