Topology Qualification Exam, Spring 2020

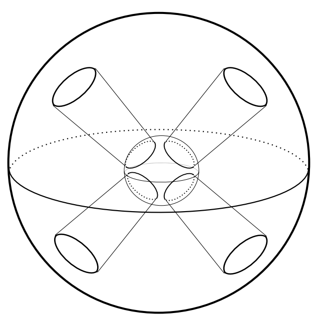
Please attempt **8 out of 9** problems and clearly mark the one you do not want us to grade.

1. Let $f : X \to Y$ be a surjective, continuous map of topological spaces.
   a) Show: if f is an open map, then it is a quotient map.
   b) Show: if f is a closed map, then it is a quotient map.

2. Show that a connected metrizable space with at least two points is uncountably infinite. (You may use without proof that every metrizable space is normal.)

3. Let (X, d_X) and (Y, d_Y) be metric spaces. An **isometric embedding** $\iota : X \to Y$ is a map such that
$$\forall x_1, x_2 \in X,\ d_Y(\iota(x_1), \iota(x_2)) = d_X(x_1, x_2).$$
   An **isometry** is a surjective isometric embedding.
   a) Show that every isometric embedding from a compact metric space to itself is an isometry.
   (You may use that a metric space is compact iff it is sequentially compact.)
   b) Show that every isometric embedding from Euclidean n-space to itself is an isometry.

4. Consider the solid S obtained by digging out the center of a 3-dimensional solid ball and 4 tunnels from the center to the boundary. What is the genus of the boundary surface $\Sigma = \partial S$? Justify your answer.



5. Let X be the topological space obtained by attaching a disk to $T^2 = S^1 \times S^1$ along the circle $S^1 \times \{p\}$ via the map $z \mapsto z^5$. Compute the fundamental group and the homology groups of X.

6. Classify the connected 2-fold covering spaces of the Klein bottle K.
   (You might want to consider K as the union of two Möbius bands.)

7. Show that every continuous map from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to $T^4 = S^1 \times S^1 \times S^1 \times S^1$ is null-homotopic.

8. Let $X = \mathbb{R}P^5/\mathbb{R}P^1$, and let $f : X \to X$ be a continuous map that is homotopic to the identity. Show that f must have a fixed point.

9. Describe the CW structure of $X = \mathbb{C}P^2 \times \mathbb{R}P^2$ and use it to compute the homology groups of X.