Topology Qualification Exam, Fall 2022

Instructions: You can assume homology groups and fundamental groups of a *point* and *wedges of spheres in all dimensions*. Everything else should be computed. Each problem is worth 10 points.

1. Consider the following equivalence relation on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$: Two points z_1 and z_2 are equivalent, i.e. $z_1 \sim z_2$, if and only if $z_1/z_2 = e^{2\pi i x}$ for some $x \in \mathbb{Q}$. Is $S^1/\sim$ Hausdorff or not? Compact or not?

2. Let Y be a compact subset of Hausdorff space X. Prove $X \setminus Y$ is open and hence Y is closed.

3. First prove that the interval $[0, 1] \subset \mathbb{R}$ is connected, and then use this to prove that a path-connected space is connected.

4. Use the classification of surfaces and the behavior of the Euler characteristic to decide which compact surfaces can be covering spaces of a genus 3 surface. Note that Euler characteristic arguments will show which surfaces *cannot* be covers of a genus 3 surface, but to show that the allowed surfaces actually are covers requires describing the covering space explicitly.

5. (a) Let T^n be the product of n copies of S^1. Describe the universal cover of T^n and the action of the deck transformation group.

   (b) Let X be a path connected, locally path connected space with $\pi_1(X, x_0)$ finite. Use lifting property of covering spaces, to show that any two continuous maps from X to T^n are homotopic.

6. Let X be the topological space constructed from $S^1 \times [0, 1]$ by identifying $(e^{i\theta}, 1) \sim (e^{3i\theta}, 0)$ for all θ. Put a CW complex structure on X and compute the fundamental group of X.

7. Use cellular homology to compute all the homology groups of $(S^1 \vee S^1) \times (S^1 \vee S^1)$.

8. Let X be the topological space obtained from $T^2 \times [0, 1]$ by identifying $T^2 \times \{0\} \coprod T^2 \times \{1\}$ to one point, where T^2 is the 2-dimensional torus. Use the long exact sequence for the pair $(T^2 \times [0, 1], T^2 \times \{0, 1\})$ to compute the homology groups of X. (If you know what the homology groups of T^2 are, you can use these without justification.)