Topology Qualifying Exam Jan. 2018

Each problem is worth 10 points.

1. Given two spaces X and Y, let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the two standard projection maps.
   (a) Show that, for any other space Z, a function $f : Z \to X \times Y$ is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.
   (b) Use this to show that the cartesian product of two path-connected topological spaces is path-connected.
2. Let $U \subset \mathbb{R}^n$ be an open set which is bounded in the standard Euclidean metric. Prove that the quotient space $\mathbb{R}^n/U$ is not Hausdorff.
3. Suppose that X is a Hausdorff topological space and that A is a subset of X. Prove that if A is compact in the subspace topology then A is closed as a subset of X.
4. (a) Write down (without proof) a presentation for $\pi_1(\Sigma_2, p)$, where Σ_2 is a closed, connected, orientable genus 2 surface and p is any point in Σ_2.
   (b) Show that $\pi_1(\Sigma_2, p)$ is not abelian by showing that it surjects onto a free group of rank 2.
   (c) Show that there is no covering space map from Σ_2 to $S^1 \times S^1$. You may use the fact that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ together with the result in part (b) above.
5. It is a fact that if X is a single point then $H_1(X) = 0$. One of the following is the correct justification of this fact in terms of the *singular chain complex*. Which one is correct and why is it correct?
   (a) $C_1(X) = 0$.
   (b) $C_1(X) \neq 0$ but $\ker \partial_1 = 0$, with $\partial_1 : C_1(X) \to C_0(X)$.
   (c) $\ker \partial_1 \neq 0$ but $\ker \partial_1 = \operatorname{image} \partial_2$, with $\partial_2 : C_2(X) \to C_1(X)$.
6. For topological spaces X, Y, the mapping cone $C(f)$ of a map $f : X \to Y$ is defined to be the quotient space $(X \times [0,1]) \sqcup Y/\sim$, where $(x, 0) \sim (x', 0)$ for all $x, x' \in X$ and $(x, 1) \sim f(x)$ for all $x \in X$. Let $\phi_k : S^n \to S^n$ be a degree k map for some integer k. Find $H_i(C(\phi_k))$ for all i.
7. Prove or disprove: Every continuous map from S^2 to S^2 has a fixed point.
8. Let D be a closed disk embedded in the torus $T = S^1 \times S^1$, and let X be the result of removing the interior of D from T. Let B be the boundary of X, i.e. the circle boundary of the original closed disk D. Compute $H_i(T, B)$ for all i.