Real analysis qualifying exam, January 2018

1. Define
$$E := \{x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N}\}.$$
Prove that $m(E) = 0$.

2. Let $f_n(x) := \frac{x}{1+x^n}$, $x \geq 0$.

   a) This sequence of functions converges pointwise. Find its limit. Is the convergence uniform on $[0, \infty)$? Justify your answer.

   b) Compute $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx$.

3. Let f be a nonnegative measurable function on $[0, 1]$. Show that
$$\lim_{p\to\infty} \left(\int_{[0,1]} f(x)^p\,dx \right)^{\frac{1}{p}} = \|f\|_\infty$$

4. Let $f \in L^2([0, 1])$ and suppose that $\int_{[0,1]} f(x)x^n\,dx = 0$ for all integers $n \geq 0$. Show that $f = 0$ a.e.

5. Suppose f_n, $f \in L^1$, $f_n \to f$ a.e. and $\int |f_n| \to \int |f|$. Show that $\int f_n \to \int f$.