Numerical Analysis Qual Exam (Spring 2020)

All problems are 10 points each.

- 1) Let A be an invertible matrix of size $n \times n$ and $\mathbf{b}$ be a vector of $n \times 1$. Define the concept of strictly diagonally dominant matrices. Define Gauss-Seidel iteration to solve $A\mathbf{x} = \mathbf{b}$. Show that if matrix A is strictly diagonally dominant, then the iterative solutions $\mathbf{x}^{(m)}, m \geq 1$ obtained by using Gaussian-Seidel's method converge to the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$.
- 2) Define the condition number for an invertible matrix A. Show that if $\|A^{-1}\|\|A - \tilde{A}\| < 1$, then the solutions to $A\mathbf{x} = \mathbf{b}$ and $\tilde{A}\hat{\mathbf{x}} = \mathbf{b}$ satisfy the following estimate

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq \frac{cond(A)\frac{\|A-\tilde{A}\|}{\|A\|}}{1 - cond(A)\frac{\|A-\tilde{A}\|}{\|A\|}}.$$

- 3) Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & -4 \end{bmatrix}.$$

  Use Gershgorin's circle theorem to analyze the locations of eigenvalues of A. Does A has complex eigenvalues? If all eigenvallues are real, where do these eigenvalues locate? Please give as good estimated eigenvalues as possible.
- 4) Let $\mathbf{v}$ be a vector of size $n \times 1$. Define its Householder's matrix $H(\mathbf{v})$. Show that $H(\mathbf{v})H(\mathbf{v}) = I_n$, the identity matrix of size $n \times n$. Explain how to transform a vector $\mathbf{v}$ to $\alpha\mathbf{e}_1$ with $\alpha = -\|\mathbf{v}\|$, where $\mathbf{e}_1$ is the unit vector with 1 in the first component and zero in the other components.
- 5) Consider a polynomial interpolation problem: given data

$$(0.1, 0.55), (0.2, 0.76), (0.4, 0.92),$$

  find polynomial P_2 satisfying $P_2(0.1) = 0.55$, $P_2(0.2) = 0.76$, and $P_3(0.4) = 0.92$ using both Lagrange interpolation formula and Newton interpolation formula. Also, give their remainder formulae.

- 6) Consider Bernstein polynomials $B_{i,n-i}(x) = \dfrac{n!}{i!(n-i)!}x^i(1-x)^{n-i}$. Show that for any continuous function $f \in C([0,1])$,

$$B_n(f) = \sum_{i=0}^{n} f(\frac{i}{n})B_{i,n-i}(x) \to f$$

  as $n \to \infty$.

- 7) Let $-\infty < \cdots < x_{-1} < x_0 < x_1 < x_2 < \cdots < \infty$ be a given sequence. Let $B_{k,i}(x) = (x_{i+k} - x_i)[x_i, x_{i+1}, \cdots, x_{i+k}](\cdot - x)_+^k$ be the normalized B-spline of order k. Find the derivative formula of $B_{k,i}$.

- 8) Explain how to use SVD to solve a least squares problem: $A\mathbf{x} = \mathbf{b}$. Also, explain how to use gradient descent method to solve the least squares problem. Finally exaplain how to speed up the gradient descent computation by using Nesterov's acceleration technique.