# Epistolary Math Tournament - Fall MMXXI

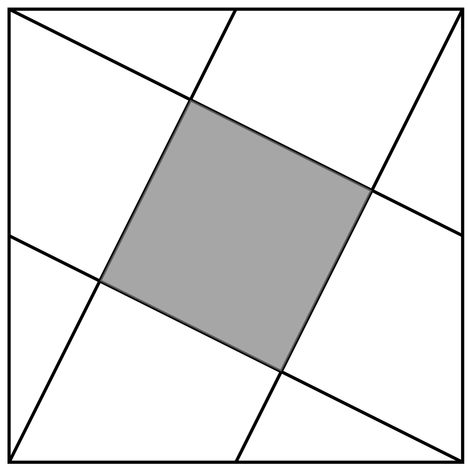
University of Georgia

Friday November 5th

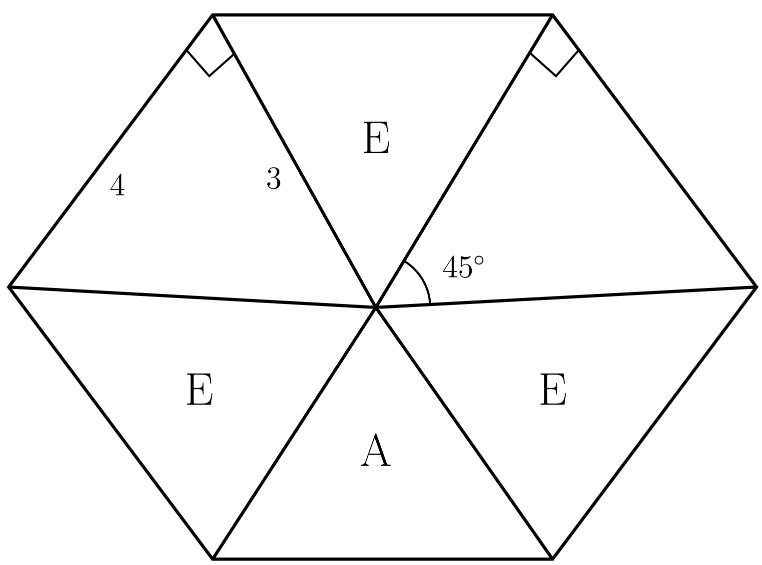
## Set 2 - Solution

## Problem 1

(a) (3pts) The diagram drawn here represents a 10 by 10 square, with four lines each joining a corner to the midpoint of a side. What is the area of the shaded region?



(b) (7pts) The hexagon shown here is not regular, and not drawn to scale. Each of the triangles labeled E is equilateral. Two side lengths and three angle measures (including the right angles) are indicated. What is the area of triangle A?



## Problem 2

(a) (3pts) You and your friend are perfect logicians. An observer announces that they have picked two positive integers $a$ and $b$ such that $a \leq b$. The observer then announces that they will give you a piece of paper with the value of $a+b$ and give your friend a piece of paper with the value of $ab$.

On your paper, you read that $a + b = 10$. You and your friend have the following conversation.

> Friend: "I know that $a \neq b$."
>
> You: "I know that $a + b < 16$."
>
> Friend: "I know the values of $a$ and $b$."

What are $a$ and $b$?

(b) (7pts) You and your friend are perfect logicians. An observer announces that they have picked two positive integers $a$ and $b$ such that $a \leq b$. The observer then announces that they will give you a piece of paper with the value of $ab$ and give your friend a piece of paper with the value of $a + b$.

On your paper, you read that $ab = 900$. You and your friend have the following conversation.

> Friend: "I know that $a \neq b$."
>
> You: "I don't know the values of $a$ and $b$."
>
> Friend: "I know that $a > 1$."

What are $a$ and $b$?

## Problem 3

(a) Consider the lines $\ell_0 : x = 0$ and $\ell_1 : x = 1$ in $\mathbb{R}^2$. We denote by $r_0$ the reflection over $\ell_0$ and by $r_1$ the reflection over $\ell_1$. So, for example, $r_0(3,0) = (-3,0)$ and $(r_1 \circ r_0)(3,0) = r_1(-3,0) = (5,0)$.

  (a) (1pt) What is $(r_0 \circ r_1 \circ r_0 \circ r_1 \circ r_0 \circ r_1 \circ r_0 \circ r_1 \circ r_0 \circ r_1)(3,0)$ [that's the composition of 10 functions alternately $r_0$ and $r_1$]?

  (b) (3pts) Given any string $s = (s_1, s_2, \ldots, s_{10})$ of 0s and 1s, we can define

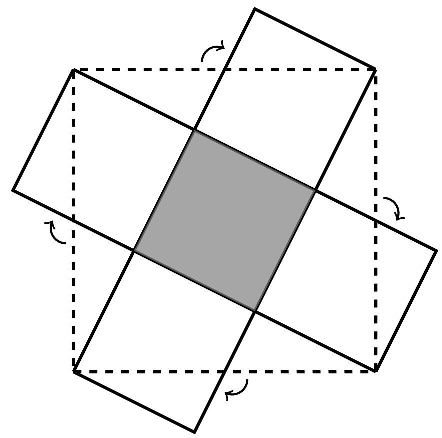
$$r_s = r_{s_1} \circ r_{s_2} \circ \cdots \circ r_{s_{10}}.$$

  [For example, if $s = (0,1,0,1,0,1,0,1,0,1)$, then $r_s$ is the function you evaluated in part $(a)$] There are $2^{10} = 1024$ possible strings $s$. How many different values can $r_s(3,0)$ take?

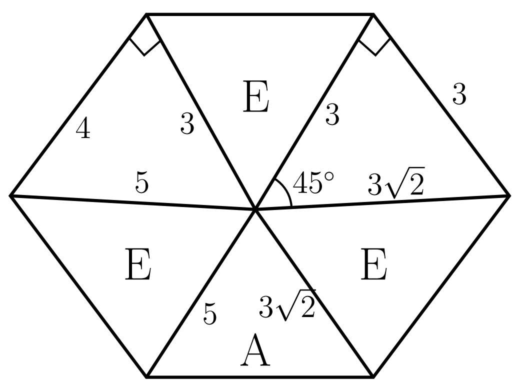
(b) (6pts) We define the functions $a(x) = 1 - x$ and $b(x) = 3 - x$. Imagine that we iteratively and randomly apply $a$ and $b$ 2021 times with 222 as initial input, what is the expected value of the final result?

**Solution 1**

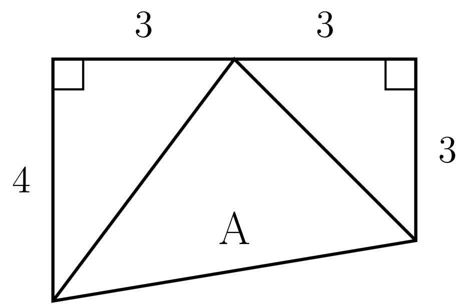
(a) By sliding a few parts around as shown, the larger square can be reimagined as five congruent smaller squares, one of which is shaded. So the shaded area is $\frac{1}{5}$ of the total area, i.e., 20.



(b) Taking advantage of the Pythagorean theorem and the equilateral triangles, we can fill in some more side lengths.




Now, remove the three equilateral triangles, and rotate the remaining three triangles together, with A in the middle:




Note that the three removed triangles contributed a total of $3 \times 60°$ to the central angle, so the remaining three central angles add up to $360° - 180° = 180°$. Thus, the resulting shape is a trapezoid with area $6 \times \left(\frac{3+4}{2}\right) = 21$.

The two right triangles have area $\frac{1}{2}(3 \times 4) = 6$ and $\frac{1}{2}(3 \times 3) = \frac{9}{2}$, so the area of A is $21 - 6 - \frac{9}{2} = \frac{21}{2}$.

## Solution 2

(a) From your information that $a + b = 10$, you know immediately that there are only 5 possibilities for the pair $(a, b)$. They are $(1, 9)$, $(2, 8)$, $(3, 7)$, $(4, 6)$, and $(5, 5)$.

The only way your friend can initially know $a \neq b$ is that the product is not a square (otherwise $a$ might equal $b$). This eliminates the pairs $(1, 9)$, $(2, 8)$, and $(5, 5)$ from consideration, leaving just $(3, 7)$ and $(4, 6)$. This means that either your friend sees $3 \times 7 = 21$ or sees $4 \times 6 = 24$.

Finally, knowing that $a + b < 16$ needs to be enough for your friend to identify $a$ and $b$. If your friend sees the product 24, then your friend would know that $(a, b)$ is one of $(1, 24)$, $(2, 12)$, $(3, 8)$, or $(4, 6)$. In this case, the information that $a + b < 16$ is not sufficient to identify $a$ and $b$. Therefore, your friend actually sees the product 21 and knows that $(a, b)$ is $(1, 21)$ or $(3, 7)$, from which the information $a + b < 16$ is enough to uniquely identify $(a, b) = (3, 7)$.

(b) The only way your friend can know initially that $a \neq b$ is that $a + b$ is odd (otherwise $a$ might equal $b$). Since $a + b$ is odd, one of $a$ and $b$ is odd while the other is even.

Your friend knows that you know one of $a$ and $b$ is odd and the other is even. Thus when you say you don't know the values $a$ and $b$, you are saying that you cannot uniquely factor the product into an even number times an odd number. Therefore your friend knows the product in not of the form $2^n$ (see $(*)$ Example below).

Finally, since your friend says they know $a > 1$, there must be some issue with $a = 1$. In particular the problem has to be that if $a = 1$, then the product would be a power of 2. Therefore, the sum your friend sees is of the form $2^n + 1$.

Let's now consider all the ways of factoring $900 = 2^2 3^2 5^2$ into an even part and an odd part. We can do this quickly by noting that the possible odd parts are the divisors of $3^2 5^2$. This gives us the possibilities of $(a, b)$ as $(1, 900)$, $(3, 300)$, $(9, 100)$, $(5, 180)$, $(15, 60)$, $(20, 45)$, $(25, 36)$, $(12, 75)$, and $(4, 225)$. Of these, only the pair $(20, 45)$ has a sum of the form $2^n + 1$, so $(a, b) = (20, 45)$.

$(*)$ Example. Suppose that you had read $ab = 16$ and that your friend opened with "I know $a \neq b$." The you would know that one of $a$ and $b$ is odd while the other is even. Since $16 = 1 \times 16$ is the only way to factor 16 into an odd number times an even number, you would already know that $(a, b) = (1, 16)$.

## Solution 3

(a) (a) Notice that $(r_0 \circ r_1)(x, y) = r_0(2 - x, y) = (x - 2, y)$, i.e. that $r_0 \circ r_1$ is a translation by 2 in the negative $x$ direction.
Using associativity,

$$(r_0 \circ r_1 \circ r_0 \circ r_1 \circ r_0 \circ r_1 \circ r_0 \circ r_1 \circ r_0 \circ r_1)(3, 0)$$

can be rewritten as

$$(r_0 \circ r_1) \circ (r_0 \circ r_1) \circ (r_0 \circ r_1) \circ (r_0 \circ r_1) \circ (r_0 \circ r_1)(3, 0)$$

i.e. the answer is $(3 - 5 \cdot 2, 0) = (-7, 0)$.

(b) Recall that $r_0 \circ r_1$ is a translation by 2 in the negative $x$ direction; similarly $r_1 \circ r_0$ is a translation by 2 in the positive $x$ direction. Notice that $(r_0 \circ r_0)(x, y) = (r_1 \circ r_1)(x, y) = (x, y)$ as indeed, reflecting a point twice over the same line returns it to its initial position. Using this observation, any string $s = (s_1 s_2 \ldots s_{10})$ is equivalent to the string obtained by removing any consecutive substring consisting of an even number of identical $r_i$'s. As we remove pairs of consecutive $r_i$'s, the sequence $s$ is thus equivalent to a sequence of the form $(r_0 \circ r_1 \ldots r_0 \circ r_1)$ or $(r_1 \circ r_0 \ldots r_1 \circ r_0)$ of length at most 10. In other words,

$$r_s(3, 0) = (r_1 \circ r_0)^n(3, 0) = (3 + 2n, 0)$$

or

$$r_s(3, 0) = (r_0 \circ r_1)^n(3, 0) = (3 - 2n, 0)$$

where $n \in \{0, 1, \ldots, 5\}$. All in all there are thus $6 + 6 - 1 = 11$ different images.

(b) Notice that $a \circ a = b \circ b$ are the identity. Also, $(a \circ b)(x) = x - 2$ and $(b \circ a)(x) = x + 2$. Consider $s$ a string of 2021 a's and b's. As $a$ and $b$ are their own inverse, any string $s$ is equivalent to the string obtained by removing any consecutive substring consisting of an even number of identical $a$'s or $b$'s. The sequence $s$ is thus equivalent to a sequence of the form $(a \circ b \circ \ldots a)$ or $(b \circ a \circ \ldots b)$ of length at most 2021. The above expressions can be decomposed as $((a \circ b)^n \circ a)(222) = (a \circ b)^n(-221) = -221 - 2n$ and $((b \circ a)^n \circ b)(222) = (b \circ a)^n(-219) = -219 + 2n$. It is easy to see that the set of strings $s$ come in pairs. Given a string $s$, we can get another string $s'$ by interchanging the a's and the b's. Now, the expected value of a fixed $s$ and $s'$ is $\frac{(-221-2n)+-219+2n}{2} = -220$. The expected value over all $s$ is therefore also $-220$.