Real Analysis Qualifying Examination

Fall 2022

The six problems on this exam have equal weighting. To receive full credit give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Let $\chi_{[0,\infty)}$ denote the characteristic function of $[0,\infty)$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that $f(x) = \chi_{[0,\infty)}(x)$ for almost every $x \in \mathbb{R}$ (with respect to Lebesgue measure).

2. Let $\{E_n\}_{n\in\mathbb{N}}$ be a countable family of Lebesgue measurable subsets of $\mathbb{R}^d$ with
$$\sum_{n=1}^{\infty} m(E_n) < \infty$$
where m denotes Lebesgue measure on $\mathbb{R}^d$ and let
$$E = \{x \in \mathbb{R}^d \,:\, x \in E_n \text{ for infinitely many } n \in \mathbb{N}\}.$$
   (a) Show that $E = \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} E_n$ and deduce that E is Lebesgue measurable with $m(E) = 0$.
   (b) Show that
$$\chi_E(x) = \limsup_{n\to\infty} \chi_{E_n}(x)$$
   for all $x \in \mathbb{R}^d$ where, for any subset A of $\mathbb{R}^d$, χ_A denote the characteristic function of A.

3. Prove that if g is continuous with compact support on $\mathbb{R}^d$, then
$$\lim_{n\to\infty} \int \left|g(n^{1/n}x) - g(x)\right| dx = 0$$
and deduce from this that if $f \in L^1(\mathbb{R}^d)$, then
$$\lim_{n\to\infty} \int \left|f(n^{1/n}x) - f(x)\right| dx = 0.$$

4. Let f be the function defined over $\mathbb{R}$ by
$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$
For a given enumeration $\{q_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let
$$F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x + q_n).$$
   (a) Prove that F is a Lebesgue integrable function on $\mathbb{R}$ and hence that the series defining F converges for almost every $x \in \mathbb{R}$.
   (b) Show, however, that this series is unbounded on every open interval, and in fact, any function G that agrees with F almost everywhere must be unbounded on every open interval.

5. Let $\{u_j\}_{j=1}^{\infty}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. Prove that the collection $\{u_{j,k}\}_{j,k=1}^{\infty}$ with
$$u_{j,k}(x,y) := u_j(x)u_k(y)$$
forms an orthonormal basis for $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

6. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$. Prove that for any integrable function $f : X \to \mathbb{C}$
$$\mu\Big(\big\{x \in X \,:\, |f(x)| \geq \frac{1}{2}\|f\|_1\big\}\Big) \geq \max\left\{\frac{\|f\|_1}{2\|f\|_\infty}, \frac{\|f\|_1^2}{4\|f\|_2^2}\right\}.$$