Algebra qualifying exam, Fall 2007

Work 8 problems including #1, which counts double. Please indicate which two problems should not be graded.

#1. a) State:
i) the first Sylow theorem,
ii) the fundamental theorem of Galois theory, and
iii) the decomposition theorem for finitely generated modules over a principal ideal domain.
b) For one of these theorems, outline the proof and give details for one significant step.

#2. Define solvability for groups, and prove every group of order p^n is solvable, where p is a prime integer.

#3. Prove every group of order 48 has a normal subgroup of index either 2 or 3.

#4. Let f be a quintic polynomial over the rational field $\mathbb{Q}$, and let E be its splitting field inside the complex numbers. If the Galois group $G(E/Q)$ is isomorphic to the symmetric group $S(5)$, prove f is irreducible over $\mathbb{Q}$.

#5. Compute the Galois group G of the splitting field L of $X^6 - 7$ over $\mathbb{Q}$. [Hint: Consider the subfields $E = \mathbb{Q}(a)$ and $F = \mathbb{Q}(b)$ where a is a primitive $6th$ root of 1, and b is a real $6th$ root of 7. Show that $L = \mathbb{Q}(a, b)$ and that $G = Aut(L/\mathbb{Q})$ is the unique non abelian semidirect product of $Aut(L/E)$ and $Aut(L/F)$. What familiar name does the group G have?]

#6. Exhibit one group in each isomorphism class of abelian groups of orders 30, 12, and 8, and explain your reasoning.

#7. Write down one matrix in each conjugacy class of those 3×3 matrices over $\mathbb{Z}/2$ with characteristic polynomial equal to each of these: $1 + X + X^2 + X^3, X + X^3$, and $1 + X^3$, and explain your reasoning.

#8. Explain why the following matrix has a Jordan form over $\mathbb{Q}$, and find a basis that puts it in Jordan form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -1 & 1 & 2 \end{bmatrix}$$

#9. i) Prove that a principal ideal domain satisfies the ascending chain condition for ideals (every strictly increasing chain of ideals is finite), and
ii) deduce that every non unit in a principal ideal domain has an irreducible factor.

#10. Prove that any two algebraic closures of $\mathbb{Q}$ are isomorphic.