Qualifying Exam in Algebra   August 2005

- Do as many problems as you can; each problem is worth 10 points. The number of problems done **completely** will also be taken into account: one correct problem is better than two half-done problems.
- **Justify** all your answers.
- Calculators are **not permitted**.

Exercise 1. (a) Prove that every group of order 45 is abelian.

(b) How many (nonisomorphic) groups of order 45 are there?

(c) Write down exactly one group from each isomorphism class.

Exercise 2. Prove that the center of a finite group of prime power order is non-trivial.

Exercise 3. (a) Define "maximal ideal" in a ring.

(b) Prove that every ring has a maximal ideal.

(c) Give an example of two different maximal ideals in the same ring.

Exercise 4. (a) Give the definition of "Euclidean domain."

(b) Prove that every Euclidean domain is a principal ideal domain.

Exercise 5. Suppose R is a commutative ring with identity where $1_R \neq 0$. Prove that the following are equivalent.

(i) R is a field;

(ii) 0 is a maximal ideal in R;

(iii) every nonzero homomorphism of rings $R \to S$ is a monomorphism.

Exercise 6. Construct (with justification) a field having 125 elements.

Exercise 7. (a) Compute the Galois group G of $x^4 - 2$ over $\mathbb{Q}$.

(b) Give a presentation of G by generators and relations.

Exercise 8. Let R be a ring and $f : M \to N$ and $g : N \to M$ be R-module homomorphisms such that $g \circ f = \mathrm{id}_M$. Show that $N \cong \mathrm{Im} f \oplus \mathrm{Ker}\, g$.

Exercise 9. Let A be a real symmetric $n \times n$ matrix, with the property that $A^k = I$ for some positive integer k. Prove that, in fact, $A^2 = I$.

Exercise 10. Determine the Jordan canonical form of the following matrix:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$