Algebra Preliminary Exam

Monday, September 16, 1996

Problems 1 and 2 are worth 20 points; the others are 12 points each.

1. State the following theorems, defining the terms in brackets:
   (a) The Sylow Theorem ("three parts") [Sylow p-subgroup]
   (b) The Spectral Theorem for a self-adjoint operator on a finite-dimensional vector space over $\mathbb{C}$; also give its interpretation for Hermitian matrices [Hermitian inner product, self-adjoint operator, Hermitian matrix, unitary matrix]
   (c) The Fundamental Theorem of Galois Theory [separable, normal, and galois extensions of fields; galois group]
   (d) The Structure Theorem for finite abelian groups, and for finitely generated modules over a PID; explain why the former is a corollary of the latter. [free module, torsion module]
   (e) The Fundamental Theorem on Symmetric Polynomials. [elementary symmetric function]

2. Quick examples: Justify your answers briefly.
   (a) Evaluate the following determinants:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 5 & 5^2 & 5^3 \end{vmatrix}$$

   (b) List the 5 isomorphism types of groups of order 8; for each, give a property (or properties) which distinguishes it from the others
   (c) Find the eigenvalues and eigenvectors of the following matrix, and determine its Jordan Canonical form:

$$\begin{pmatrix} 12 & 25 \\ -4 & -8 \end{pmatrix}$$

   (d) Let ζ be a primitive 25th root of unity. Determine the degree of the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$, and describe its Galois group.
   (e) Let G be a group with a (right) action on a group N. Define the semidirect product $N \times G$.

3. Let V and W be finite-dimensional vector spaces over a field K; let V^* be the dual space of V, and $\text{Hom}(V, W)$ be the space of K-vector space homomorphisms from V to W. Show that $V^* \otimes_K W$ is canonically isomorphic to $\text{Hom}(V, W)$.

4. Prove that every finite group of prime-power order is solvable.

5. Let R be a commutative ring with unit. Show that for any $t \in R$ which is not nilpotent, there is a prime ideal of R which does not contain any power of t. Use this to show that the intersection of all prime ideals of R is precisely the set of nilpotent elements of R [which is called the nilradical of R].

6. Let R be a Noetherian ring; prove that the polynomial ring $R[x]$ is also Noetherian.

7. (a) Explain what it means for a polynomial to be solvable by radicals.

   (b) Consider $f(x) = x^7 - 16x + 10$: find its Galois group (over $\mathbb{Q}$), and determine whether or not $f(x)$ is solvable by radicals.