Complex Analysis Qualifying Exam — Fall 2021

All problems are of equal weight. Arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

1. (10 points) Let $f(z)$ be an analytic function on $|z| < 1$. Prove that $f(z)$ is necessarily a constant if $f(\bar{z})$ is also analytic.

2. (10 points) Let $\gamma(t)$ be a piecewise smooth curve in $\mathbb{C}$, $t \in [0,1]$. Let $F(w)$ be a continuous function on γ. Show that $f(z)$ defined by

$$f(z) := \int_\gamma \frac{F(w)}{w-z} dw$$

is analytic on the complement of the curve γ.

3 (10 points) Suppose $n \geq 2$. Use a wedge of angle $\dfrac{2\pi}{n}$ to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^n} dx$$

4. (10 points) Prove that the sequence $\left(1 + \dfrac{z}{n}\right)^n$ converges uniformly to e^z on compact subsets of $\mathbb{C}$.

Hint: $e^{n \log w_n} = w_n^n$ and e^z is uniform continuous on compact subsets of $\mathbb{C}$.

5. (10 points) Assume f is an entire function such that $|f(z)| = 1$ on $|z| = 1$. Prove that $f(z) = e^{i\theta} z^n$, where θ is a real number and n a non-negative integer. [Suggestion: First use the maximum and minimum modulus theorem to show $f(z) = e^{i\theta} \displaystyle\prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z}$ if f has zeros.]

6. (10 points) Show that if $f : D(0,R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \leq M$ for some $M > 0$, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)} f(z)} \right| \leq \frac{|z|}{MR}.$$

7. (10 points) Find a conformal map from the intersection of $|z-1| < 2$ and $|z+1| < 2$ to the upper half plane.