Qualifying Exam: Complex Analysis — Fall 2018

Justify your answers and state clearly any theorem, proposition or lemma that you are applying. You should not cite examples, exercises, or problems from any source (other than this exam). Cross out the parts you do not want to be graded. Answer all seven questions.

1. Find the Laurent series expansion of $f(z) = \dfrac{1}{(z-1)(z-2)}$ valid in the annulus $1 < |z| < 2$.

2. How many roots of $z^4 + 6z + 3 = 0$ lie in the annulus $1 < |z| < 2$?

3. Write down a formula that gives a conformal equivalence between the vertical half-strip
$$\mathcal{S} = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1,\ \mathrm{Im}(z) > 0\}$$
and the upper half plane
$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

4. For each real number a, compute
$$\int_0^\infty \frac{\cos(ax)}{x^4 + 5x^2 + 4}\, dx.$$

5. **Prove or exhibit a counterexample**: If $f(z)$ is an entire function that maps every unbounded sequence to an unbounded sequence, then it is a polynomial.

6. Let $f(z)$ be a nonconstant meromorphic function on $\mathbb{C}$. A number $\omega \in \mathbb{C}$ is a period for $f(z)$ if $f(z+\omega) = f(z)$ for all z. Show that any bounded set of periods of f must be finite.

7. Let $p(z)$ be a nonconstant polynomial of even degree with all roots inside the disc $|z| < R$. Show that there is a holomorphic function $f(z)$ on the annulus $|z| > R$ with $f(z)^2 = p(z)$.