Algebra Qualifying Exam, Spring 2011

(1) Let E/F be a Galois extension with Galois group G, and suppose that $[E : F] = 28$. Suppose that there exists an intermediate field $F \subset K \subset E$ with $[E : K] = 4$ and K/F Galois.
  (a) Show that G is abelian.
  (b) Show that for **every** intermediate subfield $F \subset L \subset E$, L/F is Galois.

(2) Let G be a finite group and p a prime number. Let X_p be the set of Sylow p-subgroups of G and let n_p be the cardinality of X_p. Let $\mathrm{Sym}(X_p)$ be the permutation group on the set X_p, i.e., the set of all bijections from X_p to X_p.
  (a) Construct a homomorphism $\rho : G \to \mathrm{Sym}(X_p)$ with image a transitive subgroup (i.e. with a single orbit).
  (b) Deduce that if G is simple, $\#G \mid n_p!$.
  (c) Show that for any $1 \leq a \leq 4$ and prime power p^k, no group of order ap^k is simple.

(3) Let R be an integral domain with fraction field K. Consider the following two properties:
  (i) The intersection of all nonzero prime ideals of R is nonzero.
  (ii) There exists $x \in R$ such that $K = R[\frac{1}{x}]$.

  (a) Show that (i) implies (ii).
  (b) Let k be any field, and let R be the polynomial ring $k[t]$. Show that R *does not* satisfy (ii).

(4) Suppose that R is a principal ideal domain and $I \lhd R$ is an ideal. If $a \in I$ is an irreducible element, show that $I = Ra$.

(5) Suppose that R is a commutative ring. Show that an element $r \in R$ is not invertible if and only if it is contained in a maximal ideal.

(6) Let $K \subset L \subset M$ be a tower of finite degree field extensions. In each of the following parts, either prove the assertion or give a counterexample (with justification!).
  (a) If M/K is Galois, then L/K is Galois.
  (b) If M/K is Galois, then M/L is Galois.

(7) Let $x, y \in \mathbb{C}$ and consider the matrix $M = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix}$.
  (a) Show that $v = [0 \;\; 1 \;\; 0]^t$ is an eigenvector of M.
  (b) Compute the rank of M as a function of x and y.
  (c) Find all values of x and y for which M is diagonalizable.

(8) Suppose that V is a 6-dimensional vector space and that T is a linear transformation on V such that $T^6 = 0$ and $T^5 \neq 0$.
  (a) Find a matrix for T in Jordan canonical form.
  (b) Show that if S, T are linear transformations on a 6-dimensional vector space V which both satisfy $T^6 = S^6 = 0$ and $T^5 \neq 0 \neq S^5$. then there exists a linear transformation A from V to itself such that $ATA^{-1} = S$.