# Algebra Qualifying Examination

## Spring 2021

*Justify all the calculations and state the theorems you use in your answers. In a multi-part problem, you are free to use earlier parts in doing later parts, even if you didn't solve them.*

1. [10 pts] Let
$$A = \begin{bmatrix} 4 & 1 & -1 \\ -6 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \in M_3(\mathbf{C}).$$
   (a) [4 pts] Find the Jordan canonical form $J$ of $A$.
   (b) [5 pts] Find an invertible matrix $P$ such that $P^{-1}AP = J$. (You should not need to compute $P^{-1}$.)
   (c) [1 pt] Write down the minimal polynomial of $A$.

2. [10 pts] Let $H$ be a normal subgroup of a finite group $G$, where the order of $H$ is the smallest prime $p$ dividing $|G|$. Prove that $H$ is contained in the center of $G$.

3. [15 pts]
   (a) [5 pts] Show that every group of order $p^2$, with $p$ prime, is abelian.
   (b) [2 pts] State the three SYLOW THEOREMS.
   (c) [6 pts] Show that any group of order 4225 ($= 5^2 \cdot 13^2$) is abelian.
   (d) [2 pts] Write down exactly one representative from each isomorphism class of (abelian) groups of order 4225.

4. [10 pts] Set $f(x) = x^4 + 4x^2 + 64 \in \mathbf{Q}[x]$.
   (a) [4 pts] Find the splitting field $K$ of $f(x)$ over $\mathbf{Q}$.
   (b) [3 pts] Find the Galois group $G$ of $f(x)$.
   (c) [3 pts] Exhibit explicitly the correspondence between subgroups of $G$ and intermediate fields between $\mathbf{Q}$ and $K$.

5. [10 pts] Suppose that $f(x) \in (\mathbf{Z}/n\mathbf{Z})[x]$ is a zero divisor. Show that there is a nonzero $a \in \mathbf{Z}/n\mathbf{Z}$ with $af(x) = 0$.

6. [10 pts]
   (a) [2 pts] Carefully state the definition of NOETHERIAN for a commutative ring $R$.
   (b) [2 pts] Let $R$ be the subset of $\mathbf{Z}[x]$ consisting of all polynomials
$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$
   such that $a_k$ is even for $1 \le k \le n$. Show that $R$ is a subring of $\mathbf{Z}[x]$.
   (c) [6 pts] Show that $R$ is not Noetherian.
   Hint: Consider the ideal generated by $\{2x^k : 1 \le k \in \mathbf{Z}\}$.

7. [10 pts] Let $p$ be a prime number, and let $F$ be a field of characteristic $p$. Show that if $a \in F$ is not a $p$th power in $F$, then $x^p - a \in F[x]$ is irreducible.