ALGEBRA QUALIFYING EXAM, FALL 2020

Instructions: Complete all 8 problems. Each problem is worth 10 points. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts.

(1) (a) Using Sylow theory show that every group of order $2p$ where p is a prime is not simple.
(b) Classify all groups of order $2p$ and justify your answer. For the non-abelian group(s), give a presentation by generators and relations.

(2) Let G be a group of order 60 whose Sylow-3 subgroup is normal.
(a) Prove that G is solvable.
(b) Prove that the Sylow-5 subgroup is also normal.

(3) (a) Define what it means for a finite extension field E over F to be a Galois extension.
(b) Determine the Galois group of $f(x) = x^3 - 7$ over $\mathbb{Q}$. [Justify your answer carefully.]
(c) Find all subfields of the splitting field of $f(x) = x^3 - 7$ over $\mathbb{Q}$.

(4) Let K be a Galois extension of the field F , and let $F \subset E \subset K$ be an inclusion of fields. Let G be the Galois group of K over F, and H that of K over E. Suppose that H contains $N_G(P)$,where P is a p Sylow subgroup of G (p is a prime). Prove that $[E : F] \equiv 1 \mod p$.

(5) Consider the following 3×3-matrix.

$$B = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 2 & 3 \\ -1 & -2 & -2 \end{pmatrix}$$

(a) Find the minimal polynomial of B.
(b) Find a 3×3 matrix J in Jordan canonical form such that $B = PJP^{-1}$ where P is an invertible matrix.

(6) Let R be a ring with 1 and M a left R-module. If I is a left ideal of R, define

$$IM = \{\sum_{finite} a_i m_i | a_i \in I, m_i \in M\}$$

to be the collection of all finite sums of elements of the form am, where $a \in I$ and $m \in M$
(a) Prove that IM is a submodule of M.
(b) Let M and N be left R-modules, I a nilpotent left ideal of R and $f : M \to N$ an R-module homomomorphism. Prove that if the induced homomorphism $\overline{f} : M/IM \to N/IN$ is surjective then so is f.

(7) Let A be an $n \times n$ matrix over the real numbers $\mathbb{R}$. One can make $\mathbb{R}^n$ into a $\mathbb{R}[x]$-module by letting $f(x).v = f(A)(v)$ for $f(x) \in \mathbb{R}[x]$ and $v \in \mathbb{R}^n$. Assume that the module $\mathbb{R}^n$ has the following direct sum decomposition:

$$\mathbb{R}^n \cong \frac{\mathbb{R}[x]}{\langle (x-1)^3 \rangle} \oplus \frac{\mathbb{R}[x]}{\langle (x^2+1)^2 \rangle} \oplus \frac{\mathbb{R}[x]}{\langle (x-1)(x^2-1)(x^2+1)^4 \rangle} \oplus \frac{\mathbb{R}[x]}{\langle (x+2)(x^2+1)^2 \rangle}.$$

(a) Determine the elementary divisors and invariant factors of A.
(b) Determine the minimal polynomial of A.
(c) Determine the characteristic polynomial of A.

(8) Let A be an $n \times n$ matrix over $\mathbb{C}$ such that the group generated by A under multiplication is finite. Prove that $Tr(A^{-1}) = \overline{Tr(A)}$, where $\overline{Tr(A)}$ denotes the complex conjugate of the trace of A.