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Algebra Qualifying Exam, Fall 2013

(1) Let p and q be distinct primes.
   (a) Let $\bar{q} \in Z_p$ denote the class of q modulo p and let k denote the order of $\bar{q}$ as an element of Z_p^*. Prove that no group of order pq^k is simple.
   (b) Let G be a group of order pq. Prove G is not simple.

(2) Let G be a group of order 30.

(a) Show that G has a subgroup of order 15.

(b) Show that every group of order 15 is cyclic.

(c) Show that G is isomorphic to some semidirect product $Z_{15} \rtimes Z_2$.

(d) Exhibit three nonisomorphic groups of order 30, and prove that they are not isomorphic. **You are not required to use your answer to (c).**

(3) (a) Define **prime ideal**, and give an example of a nontrivial ideal in the ring $\mathbb{Z}$, that is not prime, and show it is not prime.
(b) Define **maximal ideal** and give an example of a nontrivial maximal ideal in the ring $\mathbb{Z}$, and show it is maximal.

(4) Let R be a commutative ring with unit element $1 \neq 0$. Recall that $x \in R$ is called **nilpotent** if $x^n = 0$ for some positive integer n.

(a) Show that the collection of all nilpotent elements of R forms an ideal.

(b) Show that if x is nilpotent, then x is contained in every prime ideal of R.

(c) Suppose that $x \in R$ is not nilpotent, and let $S = \{x^n : n \in \mathbf{N}\}$. There is at least one ideal of R disjoint from S, namely (0). By Zorn's lemma, the set of ideals disjoint from S has an element that is maximal with respect to inclusion, say I. In other words, I is disjoint from S, and if J is any ideal disjoint from S with $I \subseteq J \subseteq R$, then $J = I$ or $J = R$. Show that I is a prime ideal.

(d) Deduce from (a) and (b) that the set of nilpotent elements of R is the intersection of all of the prime ideals of R.

(5) For this problem, we let L/K denote a finite extension of fields.

(a) Define what it means for L/K to be **separable**.

(b) Show that if K is a finite field, then L/K is always separable.

(c) Give an example of a finite extension L/K that is not separable.

(6) Let K be the splitting field of $x^4 - 2$ over $\mathbf{Q}$, and let $G = \mathrm{Gal}(K/\mathbf{Q})$.

(a) Show that $K/\mathbf{Q}$ contains both $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt[4]{2})$ and has degree 8 over $\mathbf{Q}$.

(b) Let $N = \mathrm{Gal}(K/\mathbf{Q}(\sqrt{-1}))$ and $H = \mathrm{Gal}(K/\mathbf{Q}(\sqrt[4]{2}))$. Show that N is normal in G and that $NH = G$. *Hint:* What field is left fixed by NH?

(c) Show that $\mathrm{Gal}(K/\mathbf{Q})$ is generated by elements σ and τ, of orders 4 and 2 respectively, with $\tau\sigma\tau^{-1} = \sigma^{-1}$.
[This is equivalent to saying that $\mathrm{Gal}(K/\mathbf{Q})$ is the dihedral group of order 8.]

(d) How many distinct quartic subfields of K are there? Justify your answer.

(7) Let $F = \mathbf{F}_2$ and let $\bar{F}$ denote the algebraic closure of F.

(a) Show that $\bar{F}$ is not a finite extension of F.

(b) Suppose that $\alpha \in \bar{F}$ satisfies $\alpha^{17} = 1$ and $\alpha \neq 1$. Show that $F(\alpha)/F$ has degree 8.

(8) (a) What does it mean for a finite group G to be **solvable**?
(b) Prove that S_4 is solvable.

(9) (a) Suppose that T is an $n \times n$ matrix over a field F such that $T^2 = I$. Show that if the characteristic of F is not equal to 2, then T may be diagonalized, and enumerate the possibilities for the diagonal form of T.

(b) If F has characteristic 2, give an example of a matrix T such that $T^2 = I$ but T is not diagonalizable.