Ph.D Comprehensive Examination on Algebra
Fall 2006

You have three hours to complete this exam. Please write your solutions in a clear and concise fashion.

1) Let B be a complex 7×7 matrix such that $B^4 = 0$. Up to similarity, list all possible Jordan canonical forms that B can have.

2) State the three Sylow theorems. Prove that there are no simple groups of order 20.

3) Classify all groups of order 10 up to isomorphism. Justify your answer.

4) Let F be a field and $p(x)$ be a polynomial in $F[x]$. Show that the ideal $(p(x))$ in $F[x]$ is maximal if and only if $p(x)$ is irreducible over F.

5) Let k be a field and let the group $G = \mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ act on the set of $m \times n$ matrices, $\mathrm{M}_{m,n}(k)$ as follows: $(A, B).X = AXB^{-1}$ where $A, B \in G$ and $X \in \mathrm{M}_{m,n}(k)$. Exhibit with justification a subset S of $\mathrm{M}_{m,n}(k)$ which contains precisely one and only one element of each orbit under this action.

6) Consider the polynomial $p(x) = x^4 - 5 \in \mathbb{Q}[x]$.

a) Find the splitting field of $p(x)$ over $\mathbb{Q}$.

b) Compute the Galois group of $p(x)$ over $\mathbb{Q}$.

c) Are all subfields of the splitting field normal extensions over $\mathbb{Q}$? Justify your answer.

7) Let R be a Noetherian ring. Prove that $R[x]$ is also Noetherian.

8) Let M be a finitely generated module over a principal ideal domain R.

a) Let M_t be the set of torsion elements of M. Show that M_t is a submodule of M.

b) Show that M/M_t is torsion free. Is M/M_t a free module? Justify this by either stating a relevent theorem or providing a counterexample.

c) Prove that $M \cong M_t \oplus F$ where F is a free module.

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