Algebra Qualifying Examination, January 2020

Each problem is worth 10 points.

1. (a) Show that any group of order 2020 is solvable.

   (b) Give (without proof) a classification of all abelian groups of order 2020.

   (c) Describe one nonabelian group of order 2020.

2. Let H be a normal subgroup of a finite group G, where the order of H and the index of H in G are relatively prime. Prove that no other subgroup of G has the same order as H.

3. Let E be an extension field of F and $\alpha \in E$ be algebraic of odd degree over F.

   (a) Show that $F(\alpha) = F(\alpha^2)$.

   (b) Prove that α^{2020} is algebraic of odd degree over F.

4. Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$.

   (a) Define what it means for a finite extension field E of a field F to be a Galois extension.

   (b) Determine the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ for the polynomial $f(x)$. [Justify your answer carefully.]

   (c) Exhibit a subfield K in (b) such that $\mathbb{Q} \leq K \leq E$ with K not a Galois extension over $\mathbb{Q}$. Explain.

5. Let R be a ring and $f : M \to N$ and $g : N \to M$ be R-module homomorphisms such that $g \circ f = \mathrm{id}_M$. Show that $N \cong \mathrm{Im}\ f \oplus \mathrm{Ker}\ g$.

6. Let R be a ring with unity.

   (a) Give a definition for a free module over R.

   (b) Define what it means for an R-module to be torsion free.

   (c) Prove that if F is a free module, then any short exact sequence of R-modules $0 \to N \to M \to F \to 0$ splits.

   (d) Let R be a PID. Show that any finitely generated R-module M can be expressed as a direct sum of a torsion module and a free module. [You may assume that a finitely generated torsion free module over a PID is free.]

7. Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 1 \\ -16 & -16 & -2 \end{bmatrix} \in M_3(\mathbb{C})$$

   (a) Find the Jordan canonical form J of A.

   (b) Find an invertible matrix P such that $P^{-1}AP = J$. [You should not need to compute P^{-1}.]

   (c) Write down the minimal polynomial of A.

8. Let $T : V \to V$ be a linear transformation where V is a finite-dimensional vector space over $\mathbb{C}$. Prove the Cayley-Hamilton Theorem, that is $p(T) = 0$ where $p(x)$ is the characteristic polynomial of T. [You may use canonical forms.]