Topology Prelim

January, 2010

Please do all problems.

(1) Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Let X be the set of equivalence classes, endowed with the quotient topology induced by the canonical projection $\pi : \mathbb{R} \to X$. Describe, with proof, all open subsets of X with respect to this topology.

(2) If X is a topological space and $S \subset X$, define, in terms of open subsets of X, what it means for S *not* to be connected. Show that if S is not connected there are non-empty subsets $A, B \subset X$ such that $A \cup B = S$ and $A \cap \bar{B} = \bar{A} \cap B = \varnothing$ (here $\bar{A}$ and $\bar{B}$ denote closure with respect to the topology on the ambient space X).

(3) Let (X, d) be a compact metric space, and let $f : X \to X$ be an isometry (this means that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$). Show that f must be a surjection.

(4) Let B^3 be a solid ball of radius 2 centered at the origin in $\mathbb{R}^3$. Let S^1 be the unit circle in the xy-plane. Compute the homology groups of $B^3 - S^1$.

(5) Let A be a path connected subset of a path connected space X and let $i : A \subset X$ be the inclusion map. Let $p : \tilde{X} \to X$ be the universal covering map. Show that $p^{-1}(A)$ is path connected if and only if $i_{\#} : \pi_1(A, *) \to \pi_1(X, *)$ is surjective.

(6) Give a list (together with a brief explanation) of all surfaces (orientable or not, and with or without boundary) that are homotopy equivalent to the wedge of two circles.

(7) Prove or disprove: The fundamental group of a Klein bottle is an abelian group.

(8) Let X and Y be finite connected simplicial complexes, and let $f : X \to Y$ and $g : Y \to X$ be basepoint preserving maps. Show that no matter how you homotop $f \vee g : X \vee Y \to X \vee Y$, there will always be a fixed point.