Topology Preliminary Examination
Winter, 2008

Instructions: Work all problems. Give clear explanations and complete proofs. If you are asked to give an example with a certain property, be sure to prove that your example has that property.

1. Show that a compact subset of a Hausdorff space must be closed.

2. Give an example of a space that is connected but not path connected. Be sure to prove that your example is not path connected.

3. Give an example of a quotient map in which the domain is Hausdorff, but the quotient is not.

4. Give an example of a homotopy class of maps of $S^1 \vee S^1$ to itself which must have a fixed point and an example of a map of $S^1 \vee S^1$ to itself which which doesn't have a fixed point.

5. Let S be the closed orientable surface of genus 2, and let C be the commutator subgroup of $\pi_1(S, *)$. Let $\tilde{S}$ be the cover corresponding to C. Is the covering map $\tilde{S} \to S$ regular? (The term "normal" is sometimes used as a synonym for "regular" in this context.) What is the group of deck transformations? Give an example of a non-trivial element of $\pi_1(S, *)$ which lifts to a trivial deck transformation.

6. Let L be the union of the z-axis and the unit circle in the xy-plane. Compute $\pi_1(R^3 - L, *)$.

7. Let f be the map of $S^1 \times [0, 1]$ to itself defined by $f(e^{i\theta}, s) = (e^{i(\theta+2\pi s)}, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times [0, 1]$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles but not by any homotopy that is stationary on both boundary circles. (Hint: consider what f does to the path $s \to (e^{i\theta_0}, s)$ for fixed $e^{i\theta_0}$ in S^1.)

8. Let X consist of two copies of the solid torus $D^2 \times S^1$, glued together by the identity map along the boundary torus $S^1 \times S^1$. Compute the homology groups of X.