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Topology Qualifying Exam
Wednesday, January 3, 2007
9:00 am – 12:00 noon

1. Prove that the product of two compact topological spaces is compact.

2. Prove that if (X, d) is a compact metric space, $f : X \to X$ is a continuous map, and C is a constant with $0 < C < 1$ such that $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$, then f has a fixed point.

3. Let X be the quotient space of the disjoint union of two copies of the closed unit disk D^2 by the identification $x \sim h(x)$, where $h : S^1 \to S^1$ is a homeomorphism of the unit circle (the boundary of the unit disk). Prove that X is homeomorphic to the unit sphere S^2.

4. Describe the topological classification of all compact connected surfaces (2-manifolds) M without boundary with Euler characteristic $\chi(M) \geq -2$. No proof is required.

5. (a) State the Seifert-van Kampen theorem for the union of two spaces.
   (b) Use this theorem to compute the fundamental group of the Klein bottle.

6. (a) What is the definition of a *regular* (or *Galois*) covering space?
   (b) Describe a non-regular 3-sheeted covering space of the figure 8 (two circles identified at a point), and prove it is not regular.

7. Let S be a connected surface, and let U be a connected open subset of S. Let $p : \tilde{S} \to S$ be the universal cover of S. Show that $p^{-1}(U)$ is connected if and only if the homomorphism $i_* : \pi_1(U) \to \pi_1(S)$ induced by the inclusion $i : U \to S$ is onto.

8. Prove that if A is a retract of the topological space X, then for all nonnegative integers n there is a group G_n such that $H_n(X) \cong H_n(A) \oplus G_n$. (Here H_n denotes the nth singular homology group with integer coefficients.)

9. Suppose the space X is obtained by attaching a 2-cell to the torus $S^1 \times S^1$. In other words, X is the quotient space of the disjoint union of the closed unit disk D^2 and the torus $S^1 \times S^1$ by the identification $x \sim f(x)$, where S^1 is the boundary of the unit disk and $f : S^1 \to S^1 \times S^1$ is a continuous map. What are the possible homology groups of X? Justify your answer. (Again homology means singular homology with integer coefficients.)