Topology Qualifying Examination
August, 2012

Instructions: Work all problems. Give clear explanations and complete proofs.

(1) Let $X = \{(x, \sin(1/x)) | x > 0\}$ and let $Y = \{(0, y) | -1 \leq y \leq 1\}$. Show that $X \cup Y$ is not a path connected subset of $\mathbb{R}^2$.

(2) Let A denote a subset of points of S^2 that looks exactly like the capital letter A. Let Q be the quotient of S^2 given by identifying all points of A to a single point. Show that Q is homeomorphic to a familiar topological space and identify this space.

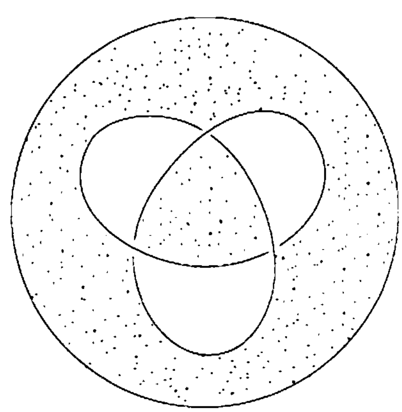
(3) Given an example of a map $f : X \to Y$ such that both spaces are connected and the map is a continuous bijection but not a homeomorphism.

(4) Let A and B be circles bounding disjoint disks in the plane $z = 0$ in $\mathbb{R}^3$. Let X be the subset of the upper half space of $\mathbb{R}^3$ that is the union of the plane $z = 0$ and a (topological) cylinder C that intersects the plane in $\partial C = A \cup B$. Compute $H_*(X)$ using the Mayer-Vietoris sequence.

(5) Use covering space theory to show that $\mathbb{Z}_2 * \mathbb{Z}$, that is, the free product of $\mathbb{Z}_2$ and $\mathbb{Z}$, has two subgroups of index 2 which are not isomorphic to each other.

(6) Let $f = id_{\mathbb{R}P^2} \vee *$ and $g = * \vee id_{S^1}$ be two maps of $\mathbb{R}P^2 \vee S^1$ to itself where $*$ denotes the constant map of a space to its base point. Show that one map is homotopic to a map with no fixed points, while the other is not.

(7) Use the classification of surfaces to identify the surface drawn below.



(8) Denote points of $S^1 \times I$ by (z, t) where z is a unit complex number and $0 \leq t \leq 1$. Let X denote the quotient of $S^1 \times I$ given by identifying $(z, 1)$ and $(z^2, 0)$ for all $z \in S^1$. Give a cell structure for X, and use it to compute $\pi_1(X, *)$ and $H_*(X)$.