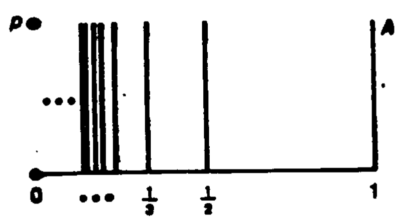
Topology Prelim
Spring 1999

Instructions: Attempt all problems. Problems 1-4, 6 and 7 are 10 points each. Problems 5 and 8 are 20 points each.

1) a) Let S be a compact space and let T be Hausdorff. Prove that any continuous bijection from S to T is a homeomorphism.

b) Show by examples that both assumptions in a) are necessary.

2) Let C be the "deleted comb space" $C = [0,1] \times \{0\} \cup (\bigcup_n \{\frac{1}{n}\} \times [0,1]) \cup \{(0,1)\} \subset \mathbb{R}^2$



Show that C is connected but not locally connected and not path connected.

3) Show by example that a quotient space of a Hausdorff space need not be Hausdorff.

4) Classify all covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^2$. Show your reasoning.

5) Let $X = S^1 \vee \mathbb{R}P^2$, the one-point union of S^1 and $\mathbb{R}P^2$.

a) Calculate the fundamental group $\pi_1(X)$ (show your work) and describe the universal covering space of X.

b) Calculate $H_*(X, \mathbb{Z})$ (show your work).

6) Show that $Free(x_1, \ldots, x_n)$, the free group on n generators, is isomorphic to a subgroup of $Free(a, b)$, the free group on two generators.

7) Let $\sum_g, g > 0$, be a closed oriented surface of genus g. Let $\pi : \sum \to \sum_g$ be a given connected k-fold cover of $\sum_g$. Given g and k, determine what topological space $\sum$ is.

8) a) Suppose $f, g \colon S^n \to S^n$ are maps with $f(x) \neq g(x)$ for all x in S^n. Show that g is homotopic to $A \circ f$ where A is the antipodal map $A(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1})$.

b) Use the statement of part a) to show that if $f : S^{2n} \to S^{2n}$ is a continuous map, then there exists an x in S^{2n} with $f(x) = x$ or there exists a y in S^{2n} with $f(y) = -y$.