·Prob

Ph.D. Preliminary Examination, March 1997

(Solve any 5 problems completely.)

1. Let $\{X_n\}$ be a sequence of independent random variables.
   (a) If $EX_n = 0$ for $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} \text{var}(X_n) < \infty$, show that $\sum_{n=1}^{\infty} X_n$ converges a.s.
   (b) State (without proof) Levy's inequality and use it to prove that $S_n = \sum_{k=1}^{n} X_k$ converges a.s. if and only if it converges in probability.
2. (a) Prove that for any r.v. X
$$E|X| = \int_0^{\infty} P(|X| \geq t)dt.$$
   (b) Given a square integrable r.v. X, show that for $\lambda \geq 0$,
$$P(X - EX \geq \lambda) \leq \frac{\sigma^2(X)}{\sigma^2(X) + \lambda^2}.$$
3. (a) State (without proof) the Levy continuity theorem regarding a sequence of characteritic functions.
   (b) Let $\{X_n\}$ be iid r.v.s with distribution $F(x)$ having finite mean μ and varivance σ^2. Let $S_n = X_1 + \cdots + X_n$. Show that
$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0, 1) \text{ in distribution as } n \to \infty.$$
4. (a) State (without proof) the Doob's maximum inequality and Kolmogorov's inequality.
   (b) Let $\mathcal{F}_n$ be a family of σ-algebras such that
$$\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$$
   and X be an integrable random variable. Show that
$$E[X|\mathcal{F}_n] \to E[X|\mathcal{F}_\infty] \text{ a.s. and in } L^1,$$
   where $\mathcal{F}_\infty = \cap_{n=1}^{\infty} \mathcal{F}_n$.
5. If $\{X_n\}$ are iid r.v.s, then $E|X_1| < \infty$ if and only if $\sum_{n=1}^{\infty} X_n \frac{\sin nt}{n}$ converges a.s. for every $t \in (-\infty, \infty)$.
6. Let $\{X_n\}$ be iid r.v.s. Then,
   (a) $n^{-1} \max_{1 \leq i \leq n} |X_i| \to 0$ in probability if and only if $nP(|X_1| > n) = o(1)$.
   (b) $n^{-1} \max_{1 \leq i \leq n} |X_i| \to 0$ a.s. if and only if $E|X_1| < \infty$.
7. (a) Given a random variable X with finite mean square. Let $\mathcal{D}$ be a σ-algebra. Show that $E[X|\mathcal{D}]$ is the minimizer of $E(X - \xi)^2$ over all $\mathcal{D}$-measurable r.v.s ξ, i.e.,
$$E(X - E[X|\mathcal{D}])^2 \leq E(X - \xi)^2$$
   for all $\mathcal{D}$-measurable r.v.s ξ.
   (b) Let $(\Omega, \mathcal{F}, P)$ denote a probability space. Suppose $f : R^n \times \Omega \to R$ is a bounded $\mathcal{B}(R^n) \times \mathcal{C}$ measurable function and X be a n-dimensional $\mathcal{D}$ measurable random variable. Assume $\mathcal{C}$ and $\mathcal{D}$ are independent. If $g(x) := Ef(x, \omega)$, then
$$g(X) = E[f(X, \omega)|\mathcal{D}], \text{ a.s.}$$