Real Analysis Qualifying Exam, January 2007.

This is a two-hour exam. The problems are weighted equally. Throughout the exam, $(\Omega, \mathcal{M}, \mu)$ denotes a measure space.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y} & x \neq y \\ f'(x) & x = y \end{cases} .$$

Prove that g is continuous.

2. Let $\{E_i\}$ be a sequence of measurable subsets of Ω such that $\sum_i \mu(E_i) < \infty$.

Set

$$S = \{\omega \in \Omega \mid \omega \in E_i \text{ for infinitely many } i.\} .$$

Show that $\mu(S) = 0$.

3. A set $\Phi \subset L^1(\Omega)$ is said to be *uniformly integrable* if, for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $f \in \Phi$ and $E \subset \Omega$ is measurable with $\mu(E) < \delta$, we have $\int_E |f| d\mu < \epsilon$.

Assume $\mu(\Omega) = 1$ and suppose $\{f_n\}_{n \in \mathbb{N}}$ is a uniformly integrable sequence of functions, such that $f_n \to 0$ almost everywhere. Prove that

$$\lim_{n \to \infty} \int_\Omega |f_n| d\mu = 0 .$$

4. Let $f \in L^\infty(\Omega) \cap L^1(\Omega)$. Show that $\lim_{p \to \infty} ||f||_p = ||f||_\infty$.
(Hint: $|f|^p = |f||f|^{p-1}$.)

5. Let I denote the unit interval, equpped with Lebesgue measure. Suppose $f \in L^2(I)$ and $F \in L^2(I \times I)$. For each $x \in I$ at which the integral exists, set

$$g(x) = \int_I f(y) F(x,y) dy .$$

Show that $g(x)$ exists almost everywhere, and that $||g||_2 \leq ||f||_2 ||F||_2$.

6. Let X be a Banach space. State the definition of the dual Banach space, X^*, and prove that X^* is complete.