Real Analysis Qualifying Exam

January, 2024

State clearly which theorem you are using. You may also quote the conclusions of one problem in this exam for another even if you do not prove the quoted ones.

Notation: m is the Lebesgue measure on the set $\mathbb{R}$ of reals, and so is dx.

1. Show that the function
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n^x}, \ x > 1$$
has a continuous derivative $f'(x)$ given by
$$f'(x) = \sum_{n=0}^{\infty} -\frac{\ln n}{n^x}, \ x > 1$$

2. Let $f : [1, \infty) \to \mathbb{R}$ be continuous, bounded function. Prove that
$$\lim_{n\to\infty} \int_1^{\infty} f(t) n t^{-n-1}\, dt = f(1).$$
[Hint: Use the special case when $f(x) = 1$.]

3. Let ϕ be a continuously differentiable function on $\mathbb{R}$ such that $\phi(x) > 0$ if $|x| < 1$ and $\phi(x) = 0$ if $|x| \geq 1$, and $\int_{\mathbb{R}} \phi(x)dx = 1$. Put $K_n(x) := n\phi(nx)$. Then (no proof required) $\int_{\mathbb{R}} K_n(y)dy = 1$ and $K_n(x) = 0$ if $|x| \geq \frac{1}{n}$. Recall that
$$f * K_n(x) = K_n * f(x) = \int_{\mathbb{R}} K_n(x - y) f(y) dy.$$
Prove the following:

(i) If $f \in L^1(\mathbb{R})$, then $f * K_n \in L^1(\mathbb{R})$ and has a continuous derivative for each n.

(ii) If $f \in L^1(\mathbb{R})$, $f * K_n$ converges to f in L^1 as $n \to \infty$.

4. Let $f \in L^1(\mathbb{R})$. Define a linear transform T_f on $L^1(\mathbb{R})$ by $T_f(g) = f * g$, $g \in L^1(\mathbb{R})$. Show that (i) $\sup\limits_{\|g\|_1 \leq 1} \|T_f(g)\|_1 = \|f\|_1$, and (ii) $T_f = 0$ if and only if $f = 0$ in $L^1(\mathbb{R})$.

5. Let f be a continuous complex valued function on $[0, 1]$. Show that
$$f([0, 1]) = \{\lambda \in \mathbb{C} : m(f^{-1}(B(\lambda, \varepsilon))) > 0 \text{ if } \varepsilon > 0\},$$
where $B(\lambda, \varepsilon) \subset \mathbb{C}$ is the open disc of radius ε centered at λ.

6. (1) Prove the following Riemann-Lebesgue identities for any $f \in L^1([-\pi, \pi])$:
$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t) \cos nt\, dt = 0, \quad \lim_{n\to\infty} \int_{-\pi}^{\pi} f(t) \sin nt\, dt = 0.$$

(2) Deduce the following: for any measurable set $E \subset [-\pi, \pi]$ and any sequence s_n of real numbers,

$$\lim_{n\to\infty} \int_E \cos^2(nt + s_n)\, dt = m(E)/2$$