Probability Theory, Ph.D Qualifying, Fall 2019

Completely solve any five problems.

1. (a) Suppose that X and Y are independent random variables with the same exponential density
$$f(x) = \theta e^{-\theta x}, \ x > 0.$$
Show that the sum $X + Y$ and the ratio X/Y are independent.

   (b) Show that the mean μ of a random variable X has the property
$$\min_c E(X - c)^2 = E(X - \mu)^2 = Var(X).$$

2. Prove for nondegenerate i.i.d. r.v.s $\{X_n\}$ that $P(X_n \text{ converges}) = 0$.

3. Let $\{X_n\}$ be a sequence of independent random variables.

   (a) If $EX_n = 0$ for $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} \text{var}(X_n) < \infty$, show that $\sum_{n=1}^{\infty} X_n$ converges a.s.

   (b) State (without proof) Levy's inequality and use it to prove that $S_n = \sum_{k=1}^{n} X_k$ converges a.s. if and only if it converges in probability.

4. Suppose that $\{X_n, \ n \geq 1\}$ is a sequence of independent identically distributed random variables with $EX_1 = 0$. Prove that
$$P\left(\frac{X_n}{n^{1/\alpha}} \to 0 \text{ as } n \to \infty\right) = 1, \alpha > 0,$$
if and only if $E|X_1|^\alpha < \infty$.

5. If $\{X_n\}$ are iid $\mathcal{L}^1$ random variables, then $\sum_{n=1}^{\infty} \frac{X_n}{n}$ converges a.s. if either (i) X_1 is symmetric or (ii) $E|X_1| \log^+ |X_1| < \infty$ and $EX_1 = 0$.

6. Prove for iid random variables $\{X_n\}$ with $S_n = X_1 + \cdots + X_n$ that
$$\frac{S_n - C_n}{n} \to 0 \text{ a.s.}$$
for some sequence of constants C_n if and only if $E|X_1| < \infty$.

7. (a) Let X_t be an $\mathcal{F}_t$–martingale and ϕ a convex function with $E|\phi(X_t)| < \infty$ for all $t \geq 0$. Show that $\phi(X_t)$ is an $\mathcal{F}_t$–submartingale.

   (b) Let X_t be a submartingale. Show that $\sup_t E|X_t| < \infty$ iff $\sup_t E(X_t)^+ < \infty$.