Probability Theory, Ph.D Qualifying, Spring 2017

Completely solve any five problems.

1. (a) If $\{X_n,\ n \geq 1\}$ are r.v.s with $\sup_{n\geq 1} E|X_n|^\beta < \infty$ for some $\beta > 0$, then $\{|X_n|^\alpha,\ n \geq 1\}$ is uniformly integrable for $0 < \alpha < \beta$.

   (b) Prove that for any r.v. X

$$E|X| = \int_0^\infty P(|X| \geq t)dt.$$

2. Given a random variable X with finite mean square. Let $\mathcal{D}$ be a σ-algebra. Show that $E[X|\mathcal{D}]$ is the minimizer of $E(X-\xi)^2$ over all $\mathcal{D}$-measurable r.v.s ξ, i.e.,

$$E(X - E[X|\mathcal{D}])^2 \leq E(X-\xi)^2$$

   for all $\mathcal{D}$-measurable r.v.s ξ.

3. Prove for nondegenerate i.i.d. r.v.s $\{X_n\}$ that $P(X_n \text{ converges}) = 0$.

4. Show that random variables $\{X_n\}$ and X satisfy $X_n \to X$ in distribution iff

$$E[F(X_n)] \to E[F(X)]$$

   for every continuous distribution function F.

5. If the independent L^1 random variables $X_1, \ldots, X_n, \ldots$ satisfy the condition

$$Var(X_i) \leq c < \infty,\ i = 1, 2, \ldots,$$

   then the SLLN holds, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - EX_i) \to 0, \text{a.s.}$$

6. (a) State (without proof) the Levy continuity theorem regarding a sequence of characteristic functions.

   (b) Let $\{X_n\}$ be iid r.v.s with distribution $F(x)$ having finite mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \text{ in distribution as } n \to \infty.$$

7. Let $X_1, X_2, \ldots$ be a sequence of independent r.v.s with $EX_i = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ and $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Show that $\phi(S_n)$ is an $\mathcal{F}_n$–submartingale for any convex ϕ provided that $E|\phi(X_n)| < \infty$ for all n.