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CIPHERING ROUND / 2 MINUTES PER PROBLEM
NOVEMBER 16, 2013

WITH SOLUTIONS

Problem 1. Ted and Valery's ages add up to 35. If Ted is twice as old as Valery was 5 years ago, how old is Ted?

Answer. 20 (years old)

Solution. Let T and V be Ted and Valery's ages. Then

$$T + V = 35, \quad T = 2(V - 5).$$

Solve carefully and you'll find $T = 20$ and $V = 15$.

Problem 2. Suppose you know $|x - 2| \leq 1$. What is the largest possible value of $|x^2 - 4|$?

Answer. 5

Solution. If $|x - 2| \leq 1$, then $-1 \leq x - 2 \leq 1$, so $1 \leq x \leq 3$. Squaring, we get $1 \leq x^2 \leq 9$, and so $-3 \leq x^2 - 4 \leq 5$. In particular, $|x^2 - 4| \leq 5$, with equality when $x = 3$.

Problem 3. Find all real values of x that satisfy the equation

$$e^x + 1 = 12e^{-x}$$

Answer. $\ln(3)$

Solution. Rewrite the equation by multiplying by e^x:

$$(e^x)^2 + e^x = 12.$$

Then treat the resulting equation as a quadratic in $u = e^x$:

$$u^2 + u - 12 = 0.$$

This has roots $u = 3$ and $u = -4$. $e^x \neq -4$ for real x, and so the only solution comes from $e^x = 3$: $x = \ln(3)$.

Problem 4. Paul took 1 minute to solve the first problem on an 8 question math test, and each successive problem took twice as long as the preceding problem. Assuming he took a 1 minute break between problems, how long did it take Paul to complete the test?

Answer. 262 (minutes)

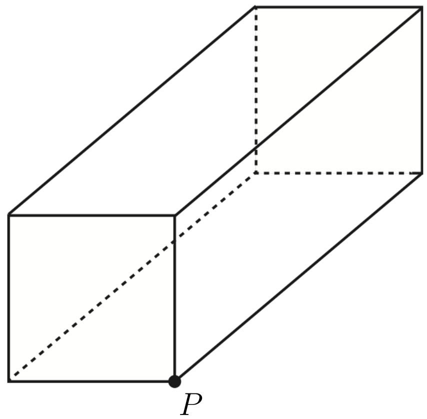
Solution. Paul's times on the 8 questions form a geometric sequence $1, 2, 2^2, \ldots, 2^7$. These sum to $\frac{2^8-1}{2-1} = 255$. Add in 7 minutes for breaks, and Paul took a total of 262 minutes.

Problem 5. Find the smallest integer k greater than 1 such that k divided by i has remainder 1 for all i in the set $\{2, 3, 4, 5, 6, 7, 8\}$.

Answer. 841

Solution. If k is the smallest such integer, then $k-1$ is the smallest integer divisible by $2, 3, 4, 5, 6, 7$, and 8, i.e., their least common multiple. The LCM is $3 \cdot 5 \cdot 7 \cdot 8 = 840$, so $k = 841$.

Problem 6. We define the distance between two points on the surface of a $1 \times 1 \times 2$ rectangular box to be the length of the shortest path (on the surface of the box) which joins them. With this definition, the circle of radius 1 centered at P is the set of all points which are a distance of 1 from P. What is the circumference of this circle?



Answer. $3\pi/2$

Solution. The circle consists of three circular arcs, one on each of the rectangular faces meeting at P. Each of the arcs is a quarter circle, so each has length $\pi/2$. So the circumference is $3\pi/2$.

Problem 7. Find the sum of the solutions of

$$x^4 - 76x^3 + 962x^2 - 2900x + 2013 = 0$$

Answer. 76

Solution. If a polynomial $p(x)$ factors as

$$p(x) = (x - a)(x - b)(x - c)(x - d),$$

then

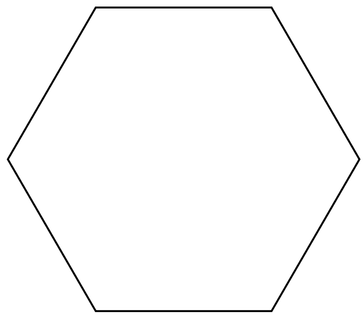
$$p(x) = x^4 - (a + b + c + d)x^3 + \cdots + abcd.$$

So the sum (or product) of the roots can be seen in the coefficients. In this case, the sum of the roots is 76.

In fact, this polynomial factors as

$$(x - 1)(x - 3)(x - 11)(x - 61).$$

Problem 8. If three vertices of a regular hexagon are chosen at random, what is the probability that they are the vertices of a right triangle?



Answer. 3/5 or 60%

Solution. Three vertices form a right triangle if and only if they include two antipodal points. There are 3 pairs of antipodal points, and any of these can be paired with 4 other points, so there are 12 right triangles. There are $\binom{6}{3} = 20$ total triangles. So the probability is $\frac{12}{20} = \frac{6}{10} = \frac{3}{5} = 60\%$.

Problem 9. How many degree 4 monomials are there in the variables w, x, y, z? A degree 4 monomial is a term $w^a x^b y^c z^d$ such that a, b, c and d are integers with $0 \leq a, b, c, d \leq 4$ and $a + b + c + d = 4$.

Answer. 35

Solution. We need to partition 4 into a sum of exactly 4 nonnegative integers. Here's one way to think of this: Start with 7 "placeholders", 4 for the integers and 3 for the plus signs:

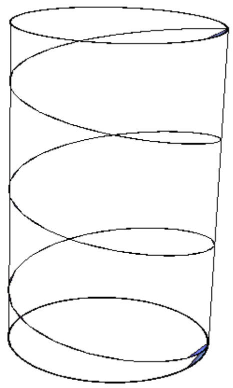
$$\underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}\ \underline{\qquad}.$$

Any choice of placement for the 3 + signs determines a partition; for example,

$$\underline{\qquad}\ \underline{\quad + \quad}\ \underline{\qquad}\ \underline{\quad + \quad}\ \underline{\quad + \quad}\ \underline{\qquad}\ \underline{\qquad}$$

corresponds to $1 + 1 + 0 + 2$, which in turn corresponds to the monomial wxz^2. So there are $\binom{7}{3} = 35$ such monomials.

Problem 10. A string will wrap around the base of a certain cylinder exactly 5 times. If instead the same string spirals tightly to the top, it goes around the cylinder exactly 3 times. If the radius of the cylinder is 1, what is the height of the cylinder? Your answer should be written as an integral multiple of π.



Answer. 8π

Solution. If you "unwrap" the cylinder 3 times, you'll see a rectangle whose base is 3 times the circumference of the cylinder — $3(2\pi r) = 6\pi$ — and whose height h is the unknown we seek. Since the string would wrap 5 times around the base, its length is $5(2\pi r) = 10\pi$. This is a diagonal of the unwrapped rectangle, and so $(6\pi)^2 + h^2 = (10\pi)^2$. Thus, $h = 8\pi$.

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