Qualifying Examination Problems, Fall, 2024

There are 8 problems from three subjects. If you complete 5 of them correctly, you have a strong pass of the examination. If you complete 4 of them correctly and some partial credits with total points more than 50 points over the 80 points, you get a master pass.

Numerical Linear Algebra

- 1.(10 points) Consider a square matrix A.

  (4 points) Describe how to use the Gaussian elimination algorithm (without pivoting) to obtain the LU decomposition of A, provided that the pivot entry is never zero.

  (3 points) Suppose A is invertible, and there exists an LU decomposition of A. Prove that the LU decomposition of A in which L has diagonal entries all equal to 1 is unique.

  (3 points) Describe how to use the LU decomposition to solve a linear system $A\mathbf{x} = \mathbf{b}$. Explain the advantage of using LU decomposition to solve many linear systems $A\mathbf{x} = \mathbf{b}_1, \ldots, A\mathbf{x} = \mathbf{b}_m$ with the same coefficient matrix, comparing to solving them separately by Gaussian elimination.

- 2(10 points) Consider the Gauss-Seidel method for solving a linear system $A\mathbf{x} = \mathbf{b}$.

  (3 points) Formulate the method in the form of

  $$Q\mathbf{x}^{(k)} = (Q - A)\mathbf{x}^{(k-1)} + \mathbf{b}$$

  where the splitting matrix Q should be specified.

  (3 points) Define the spectral radius $\rho(B)$ of a square matrix B. Also define a square matrix B is diagonally dominant.

  (4 points) Suppose A is diagonally dominant. Prove that $\rho(I - Q^{-1}A) < 1$ where Q is as in the Gauss-Seidel method.

- 3(10 points) Consider the power method for finding eigenvalues of a square matrix A.

  (3 points) Formulate the method.

  (3 points) Suppose A has eigenvalues with distinct modulus. Which eigenvalue does the method find and why? Please give a heuristic explanation.

  (4 points) Explain how to find the eigenvalue of A with the smallest modulus? Give some heuristic reasons that your method will find it.

Numerical Approximation

- 4(10 points) Given distinct points $x_0, \ldots, x_n \in \mathbb{R}$, and function values $f(x_0), \ldots, f(x_n)$.

  (3 points) State the definition of the polynomial interpolation of f at $x_0, \ldots, x_n$. Given an explicit formula for the polynomial interpolation in the Lagrange form.

  (3 points) Prove the uniqueness of the polynomial interpolation (i.e., prove the uniqueness of a polynomial satisfying the characterizing properties of the polynomial interpolation).

  (4 points) Let $f(x) = \sin 2x$. Give a good estimate for the L^∞ error on $[0, 1]$ of its polynomial interpolation at base points $0, \frac{1}{3}, \frac{2}{3}, 1$. That is, you do not need to find the optimal estimate. An estimate that shows that the error is small is a good estimate.

- 5(10 points) Consider the numerical approximation of an integral $\int_{-1}^{1} f(x)w(x)dx$, where $w(x) = 1 + x^2$ is the weight function.

  (5 points) Define orthogonal polynomials over $[-1, 1]$ with weight function $w(x) = 1 + x^2$. Starting with $P_0(x) = 1$, calculate the orthogonal polynomials P_1, P_2.

(3 points) Use the above results to give the explicit formula for the Gauss quadrature for the integral $\int_{-1}^{1} f(x)w(x)dx$, $w(x) = 1 + x^2$ with two base points. That is, the quadrature formula with points and coefficients should be stated explicitly.

(2 points) For the quadrature stated above, for what degrees of polynomials is the quadrature exact?

- 6(10 points) Consider the numerical approximation of $f'(x)$ for a given function $f(x)$ using its point values.

  (4 points) Derive the numerical differentiation formula with optimal accuracy order using $f(x), f(x+h), f(x+2h)$.

  (4 points) Use the Richardson extrapolation to improve its accuracy order. What is the accuracy order of the resulting method?

  (2 points) For the last quadrature you derived, how many point values are used? Is it the numerical differentiation formula with optimal accuracy using these point values?

Numerical Differential Equations

- 7(10 points) Consider the ODE

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

  (5 points) Write the formula for a second order Runge-Kutta method and write its Butcher table.

  (5 points) Prove that this method is at least second order accurate.

- 8(10 points) Consider the second order ODE

$$x''(t) = -2x'(t) - 3(x(t))^3, \quad x(0) = -1, \quad x'(0) = -3$$

  (3 points) Reformulate it as an initial value problem of a system of first order ODEs.

  (7 points) Apply the second order Adams-Bashforth method with time step $h = 0.2$ and give explicit formulas for the iterations. Then use the forward Euler method to initiate the iteration and calculate x_1, v_1. Then start the second order Adams-Bashforth method and calculate x_2.