Numerical Analysis Qualifying Examination, Spring 2024

Each problem is worth 10 points.

1. (a) (3 points) Describe in detail the *Jacobi iterative method* for solving a system of linear equations $A\mathbf{x} = \mathbf{b}$.
   (b) (2 points) Define what it means for a matrix to be *strictly diagonally dominant.*
   (c) (5 points) Show that the Jacobi iterative method converges when the matrix A is strictly diagonally dominant.

2. (a) (2 points) Define the *QR factorization* of a matrix A.
   (b) (4 points) Give a procedure for computing the QR factorization of A.
   (c) (4 points) Suppose that A is a tridiagonal 3×3 matrix with QR factorization $A = QR$. Show that RQ is also a tridiagonal matrix.

3. (a) (3 points) Define the C^2 cubic spline space $\mathcal{S}_3^2(\triangle)$ over a partition

$$\Delta = \{a = x_0 < x_1 < \cdots , x_n = b\}$$

   of the interval $[a, b]$.
   (b) (4 points) Given values $f(x_0), \ldots, f(x_n)$, let S_f be the natural cubic interpolating spline over the interval $[a, b]$, i.e.

$$S_f \in \mathcal{S}_3^2(\triangle), \quad S_f(x_i) = f(x_i), i = 0, \cdots, n. \tag{1}$$

   Suppose this cubic spline is given by

$$S_f|_{[x_i, x_{i+1}]} = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, i = 0, \cdots, n - 1.$$

   Derive relations among a_i, b_i, c_i, d_i and $a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$ that ensure that $S_f \in \mathcal{S}_3^2(\triangle)$.
   (c) (3 points) Show that these conditions may be reduced to a system of linear equations in the c_i, where the associated matrix is invertible.

4. Recall that the *Legendre polynomials* $P_0, (x), P_1(x), \ldots$ are the polynomial functions determined by the conditions

$$\deg P_n(x) = n, \quad P_n(1) = 1, \quad m \neq n \implies \int_{-1}^{1} P_m(x) P_n(x)\, dx = 0.$$

   (a) (3 points) Given a function $f(x)$, explain how to use the $P_n(x)$ for numerical approximation of $\int_{-1}^{1} f(x)dx$.
   (b) (3 points) What is the advantage of this process (known as *Gauss quadrature*) over the Newton Cotes formula?
   (c) (4 points) Let $G_n(f)$ be the Gauss quadrature using the orthogonal polynomial of degree n. Show that $G_n(f) \to \int_{-1}^{1} f(x)dx$ as $n \to \infty$ for any continuous function f.

5. Consider the numerical differentiation formula

$$f'(x) \approx \frac{1}{h}\left(-\frac{1}{4}f(x-h) - \frac{3}{4}f(x) + \frac{5}{4}f(x+h) - \frac{1}{4}f(x+2h)\right)$$

   (a) (4 points) Determine the order of accuracy of this approximation.
   (b) (4 points) Use Richardson extrapolation to improve its order of accuracy . What is the order of accuracy of the resulting method?

(c) (2 points) Explain why this formula cannot give a good approximation of $f'(x)$ if h is taken extremely small.

6. Consider the ODE

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

   (a) (4 points) Write the formula for Heun's method and write its Butcher table.

   (b) (6 points) Prove that Heun's method is at least second order accurate.

7. Consider the second order ODE

$$x''(t) = -3x'(t) - 2(x(t))^3, \quad x(0) = 1, \quad x'(0) = -2$$

   (a) (3 points) Reformulate it as an initial value problem of a system of first order ODEs.

   (b) (7 points) Apply the second order Taylor method with time step h and give explicit formulas for the iterations. Calculate the first iteration x_1 for $h = 0.1$.

8. For the ODE $x'(t) = f(t, x(t))$, consider the general linear multistep method

$$a_k x_i + \cdots + a_0 x_{i-k} = h(b_k f_i + \cdots + b_0 f_{i-k})$$

   where $f_j := f(t_j, x_j)$ and $a_0, \ldots, a_k, b_0, \ldots, b_k$ are coefficients with $a_k \neq 0$.

   (a) (7 points) Let p be a given positive integer. Derive the linear relations on the coefficients a_i, b_j that are necessary and sufficient condition for this method to have at least p-th order accuracy.

   (b) (3 points) Consider the case $k = 2$ and $(b_0, b_1, b_2) = (0, 0, 1)$. Determine the coefficients a_0, a_1, a_2 so that the above method has the highest possible order of accuracy .