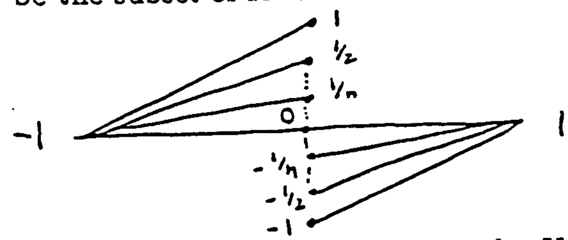
May 1995

Topology Prelim

1. Let X be the subset of $\mathbb{R}^2$ shown below. Is X contractible?

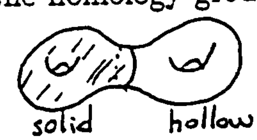


2. Let X and Y be metric spaces and let $f : X \times I \longrightarrow Y$ be a homotopy. Define $D(t)$, the diameter at time t, to be $\sup_{a,b \in X}\{d_Y(f(a,t), f(b,t))\}$,
   a) Show that if X is compact then D is continuous.

   b) Show that if we only assume Y is compact then D may not be continuous.

3. Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ iff $x - y \in \mathbb{Q}$. Let $\mathbb{R}/\mathbb{Q}$ be the set of equivalence classes and give $\mathbb{R}/\mathbb{Q}$ the quotient topology with respect to the natural map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Q}$. Is $\mathbb{R}/\mathbb{Q}$ compact? Is $\mathbb{R}/\mathbb{Q}$ Hausdorff?

4. Compute the homology groups of X where X is the "half-solid" surface of genus 2, i.e.



5. Prove or disprove: every map $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2$ has a fixed point.

6. Define the natural map $\pi_1(X, *) \longrightarrow H_1(X)$. Show that it can have a kernel.

7. Show that there is a map $f : S^1 \times S^1 \to S^2$ of degree 2, that is, such that $f_* : H_2(S^1 \times S^1) \longrightarrow H_2(S^2)$ maps a generator to twice a generator.

8. Show that there is a space X with $\pi_1(X, *) = \mathbb{Z}_n$.

9. Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a convering map and let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a map. Assume Y is path-connected and locally path-connected. State and prove necessary and sufficient conditons for the existence of a map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ satisfying $p \circ \tilde{f} = f$.