Carl F. Kossack Calculus Prize Examination (Solutions)

April 6, 2024

1. [10 pts] Determine the following limit.

$$\lim_{x\to-\infty}(\sqrt{x^2+x}-\sqrt{x^2-x})$$

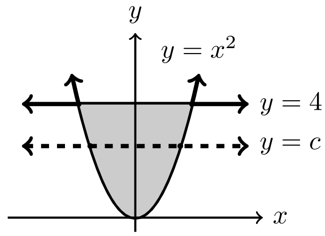
First, rationalize this by multiplying and dividing the conjugate. Then factor $\sqrt{x^2}$ out of the denominator...

$$\begin{aligned}\lim_{x\to-\infty}(\sqrt{x^2+x}-\sqrt{x^2-x}) &= \lim_{x\to-\infty}(\sqrt{x^2+x}-\sqrt{x^2-x})\cdot\frac{(\sqrt{x^2+x}+\sqrt{x^2-x})}{(\sqrt{x^2+x}+\sqrt{x^2-x})}\\ &= \lim_{x\to-\infty}\frac{(x^2+x)-(x^2-x)}{(\sqrt{x^2+x}+\sqrt{x^2-x})}\\ &= \lim_{x\to-\infty}\frac{2x}{\sqrt{x^2}\cdot(\sqrt{1+1/x^2}+\sqrt{1-1/x^2})}\end{aligned}$$

Because x approaches *negative* infinity, so that $x<0$, we have $\sqrt{x^2}=\sqrt{|x|^2}=|x|=-x$, so we obtain

$$\lim_{x\to-\infty}\frac{2x}{(-x)(\sqrt{1+1/x^2}+\sqrt{1-1/x^2})} = \boxed{\frac{-2}{\sqrt{1+0}+\sqrt{1-0}}} = \frac{-2}{2} = -1$$

2. [10 pts] Consider the region enclosed by $y = x^2$ and $y = 4$, pictured below. Determine the value of c so that $y = c$ divides the region into two parts of the same area.



There are two approaches, one that uses dx integrals and one that uses dy integrals.

We'll show the approach using dy integrals.

Note that $y = x^2$ becomes $x = \pm\sqrt{y}$, so the total area shaded area is

$$\text{Total} = \int_{y=0}^{y=4} (\sqrt{y}) - (-\sqrt{y})\,dy = \int_0^4 2\sqrt{y}\,dy = 2 \cdot \frac{2}{3} y^{3/2}\bigg|_0^4 = \frac{4}{3} \cdot 4^{3/2}$$

The area under the line $y = c$ is the same as over the line $y = c$, so each of these is half of the total area. Namely,

$$\frac{1}{2}\text{Total} = \frac{2}{3} \cdot 4^{3/2} = \int_{y=0}^{y=c} 2\sqrt{y}\,dy = \frac{4}{3} y^{3/2}\bigg|_0^c = \frac{4}{3} c^{3/2}$$

We now solve for c:

$$\frac{4}{3} c^{3/2} = \frac{2}{3} 4^{3/2} = \frac{2}{3} \cdot 8 = \frac{16}{3} \qquad \Leftrightarrow \qquad c^{3/2} = 4 \qquad \Leftrightarrow \qquad \boxed{c = 4^{2/3}} = 16^{1/3}$$

Points earned: ______
out of a possible 10 points

3. [13 pts] Compute the following integral, showing all work.

$$\int_0^1 \frac{dx}{(x^2+1)^3}$$

Start with the trigonometric substitution $x = \tan(\theta)$ aka $\theta = \arctan(x)$:

$$x = \tan(\theta) \qquad dx = \sec^2(\theta)\,d\theta \qquad x^2+1 = \tan^2(\theta)+1 = \sec^2(\theta)$$

The limits also change, giving us:

$$\int_{\theta=\arctan(0)}^{\theta=\arctan(1)} \frac{\sec^2(\theta)\,d\theta}{(\sec^2(\theta))^3} = \int_0^{\pi/4} \frac{1}{\sec^4(\theta)}\,d\theta = \int_0^{\pi/4} \cos^4(\theta)\,d\theta$$

Write this as $(\cos^2(\theta))^2$ and use the half-angle identity:

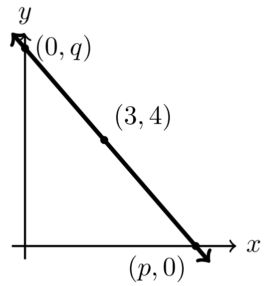
$$\int_0^{\pi/4} \left[\frac{1}{2}(1+\cos(2\theta))\right]^2 d\theta = \frac{1}{4}\left[\int_0^{\pi/4} 1 + 2\cos(2\theta) + \cos^2(2\theta)\,d\theta\right]$$

Use the half-angle identity one more time with $\cos^2(2\theta)$:

$$\begin{aligned}
& \frac{1}{4}\left[\int_0^{\pi/4} 1 + 2\cos(2\theta) + \frac{1}{2}\left(1+\cos(4\theta)\right)\,d\theta\right] \\
= & \frac{1}{4}\left[\theta + \sin(2\theta) + \frac{1}{2}\left(\theta + \frac{1}{4}\sin(4\theta)\right)\right]\Bigg|_0^{\pi/4} \\
= & \boxed{\frac{1}{4}\left[\frac{\pi}{4} + \sin(\pi/2) + \frac{1}{2}\cdot\frac{\pi}{4} + \frac{1}{8}\cdot\sin(\pi)\right] - \frac{1}{4}\left[0 + \sin(0) + \frac{1}{2}\cdot 0 + \frac{1}{8}\sin(0)\right]} \\
= & \frac{1}{4}\left(\frac{3\pi}{8}+1\right) = \frac{3\pi}{32} + \frac{1}{4}
\end{aligned}$$

Points earned: ______
out of a possible 13 points

4. [15 pts] A straight line passes through the point $(3,4)$, intersecting the x-axis at $x = p$ and the y-axis at $y = q$ (with $p, q > 0$). Determine the minimum possible value of $p + q$. Justify your answer.



One method involves realizing the slope of this line can be written in two ways:
it can be written through $(0,q)$ and $(3,4)$, or through $(3,4)$ and $(p,0)$, so

$$m = \frac{q-4}{0-3} \text{ and } m = \frac{0-4}{p-3}$$

Setting these equal lets us solve q in terms of p:

$$\frac{q-4}{-3} = \frac{-4}{p-3} \quad \Leftrightarrow \quad q - 4 = \frac{12}{p-3} \quad \Leftrightarrow \quad q = \frac{12}{p-3} + 4$$

Therefore, our objective is to minimize $p + q$, which is

$$F(p) = p + \left(\frac{12}{p-3} + 4\right) = p + 12(p-3)^{-1} + 4$$

Notice that we need $(p, 0)$ to the right of $(3, 4)$ for this problem to make sense, so $p > 3$ (and hence $(p - 3) > 0$ in the denominator of $12/(p - 3)$). The domain of $F(p)$ is the open interval $(3, \infty)$.

Compute the derivative and set it to 0 to get critical points. (We also make a common denominator.)

$$F'(p) = 1 - 12(p-3)^{-2} = 1 \cdot \frac{(p-3)^2}{(p-3)^2} - \frac{12}{(p-3)^2} = \frac{(p-3)^2 - 12}{(p-3)^2} = 0$$

This shows the derivative does not exist at $p = 3$ (not in domain anyway), and $F'(p) = 0$ when $(p - 3)^2 = 12$, so $p - 3 = \pm\sqrt{12}$ and $p = 3 \pm \sqrt{12}$. However, because we need $p > 3$, only $p = 3 + \sqrt{12}$ makes sense.

Because $F(p)$ only has a single critical point, if we can show it is a local minimum, then it is also the absolute minimum. One method is to use the Second-Derivative Test:

$$F''(p) = 24(p-3)^{-3} = \frac{24}{(p-3)^3}$$

Therefore, $F''(3 + \sqrt{12}) = 24/(\sqrt{12})^3 > 0$, indicating we have a local min.

The smallest value of $p + q$ occurs with

$$\boxed{p = 3 + \sqrt{12}} \qquad \boxed{q = \frac{12}{p-3} + 4 = \frac{12}{\sqrt{12}} + 4 = \sqrt{12} + 4}$$

Points earned: ______
out of a possible 15 points

5. [15 pts] Let $g(x)$ be a **polynomial of degree** 4 with the following properties:

- The graph of $g(x)$ has critical points at $x = 0$, $x = -3$, and $x = 4$.
- At $x = 1$, the tangent line to $g(x)$ is $y = 6x + 25$.

Determine a possible formula for $g(x)$, with justification.

(**HINT:** First write $g'(x)$ in factored form with an unknown leading coefficient.)

If $g(x)$ is a polynomial of degree 4, then $g'(x)$ is a polynomial of degree 3. Furthermore, the three critical points are roots of the derivative, so $g'(0) = g'(-3) = g'(4) = 0$. This means that $(x-0)$, $(x+3)$, and $(x-4)$ are factors of $g'(x)$. Therefore,

$$g'(x) = Kx(x+3)(x-4) \quad \text{for some constant } K$$

Because we have a tangent line of $y = 6x + 25$ at $x = 1$, this means the slope at $x = 1$ is 6, so $g'(1) = 6$. Plugging that into our $g'(x)$ formula:

$$g'(1) = 6 = K \cdot 1(4)(-3) = -12K \qquad \Leftrightarrow \qquad K = \frac{6}{-12} = \frac{-1}{2}$$

Now we expand out $g'(x)$ and integrate to determine $g(x)$ up to a constant C:

$$\begin{aligned} g'(x) &= \frac{-1}{2}x(x+3)(x-4) = \frac{-x}{2}(x^2 - x - 12) = \frac{-1}{2}x^3 + \frac{1}{2}x^2 + 6x \\ g(x) &= \int g'(x)\,dx = \frac{-1}{8}x^4 + \frac{1}{6}x^3 + 3x^2 + C \end{aligned}$$

Finally, the tangent line of $y = 6x + 25$ at $x = 1$ also tells us that $(1, 6+25) = (1, 31)$ is on $g(x)$. Plugging this in to the equation for $g(x)$, we get

$$g(1) = 31 = \frac{-1}{8} + \frac{1}{6} + 3 + C$$

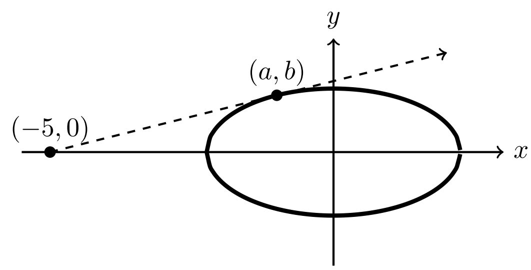
Solving for C, we obtain the final answer

$$\boxed{g(x) = \frac{-1}{8}x^4 + \frac{1}{6}x^3 + 3x^2 + 31 - 3 - \frac{1}{6} + \frac{1}{8}} = \frac{-1}{8}x^4 + \frac{1}{6}x^3 + 3x^2 + \frac{671}{24}$$

Points earned: ______
out of a possible 15 points

6. [12 pts] A spotlight located at the point $(-5, 0)$ shines **upward** at an ellipse $x^2 + 4y^2 = 5$. The light beam passes tangent to the ellipse through a point (a, b). Determine that point.

(**HINT:** Find the slope of the line in two different ways, one of which uses implicit differentiation.)



One way of obtaining the slope of the tangent line through $(-5, 0)$ and (a, b) is

$$m = \frac{b - 0}{a - (-5)} = \frac{b}{a+5}$$

The other way is to use implicit differentiation of the ellipse:

$$\begin{aligned} \frac{d}{dx}\left[x^2 + 4y^2\right] &= \frac{d}{dx}[5] \\ 2x + 8y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-2x}{8y} = \frac{-x}{4y} \end{aligned}$$

Evaluated at the point (a, b), this means $m = -a/(4b)$.

Setting those formulas for m quald, and cross-multiplying, we obtain

$$\frac{b}{a+5} = \frac{-a}{4b} \quad \Leftrightarrow \quad 4b^2 = -a(a+5) = -a^2 - 5a \quad \Leftrightarrow \quad a^2 + 4b^2 = -5a$$

However, the ellipse equation with (a, b) plugged in yields $a^2 + 4b^2 = 5$. Therefore,

$$a^2 + 4b^2 = 5 \text{ and } a^2 + 4b^2 = -5a \text{ so } -5a = 5$$

Therefore, $a = -1$. Put this in the ellipse equation to solve b:

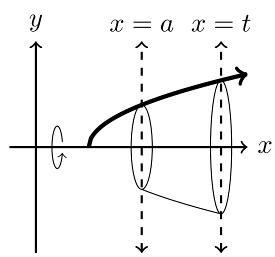
$$(-1)^2 + 4b^2 = 5 \qquad \Leftrightarrow \qquad 4b^2 = 4 \qquad \Leftrightarrow \qquad b = \pm 1$$

Since this point is in Quadrant 2 (with light shining upward), $b > 0$. We have $\boxed{(a, b) = (-1, 1)}$.

Points earned: ______
out of a possible 12 points

7. [10 pts] Consider the region enclosed by some positive continuous function $y = f(x)$ and the x-axis between the lines $x = a$ and $x = t$ (where a is constant and $t > a$). When this region is rotated around the x-axis, the volume of revolution is $V(t) = t^2 - at$. Determine a formula for $f(x)$, with justification. (The answer may use a in it.)

(**HINT:** Which part of the Fundamental Theorem can be used here?)



This solid of revolution obtains disk cross-sections with radius $R = f(x)$.

The volume of revolution on the interval $[a, t]$ is

$$V(t) = t^2 - at = \int_a^t \pi f(x)^2 \, dx$$

Since $f(x)$ is continuous and a is constant, we may use the first part of the Fundamental Theorem and differentiate with respect to t to get

$$\frac{d}{dt}(t^2 - at) = \frac{d}{dt}\left[\int_a^t \pi f(x)^2 \, dx\right] \qquad \Rightarrow \qquad 2t - a = \pi f(t)^2$$

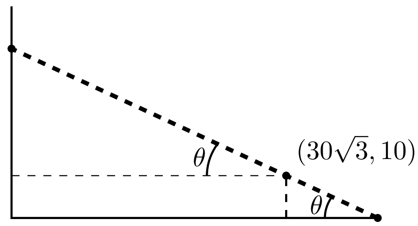
Solving for $f(t)$, taking the positive square root since $f(t) > 0$, yields

$$\boxed{f(t) = \sqrt{\frac{2t - a}{\pi}}}$$

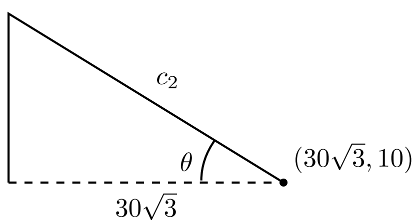
Points earned: ______
out of a possible 10 points

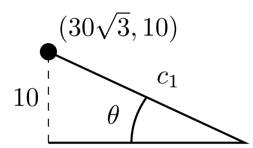
8. [15 pts] In the following picture, imagine an infinitely-tall wall. A lamppost is 10 feet tall and is located $30\sqrt{3}$ feet to the right of the wall. We want to place a ladder (drawn in thick dashes along the hypotenuse) with its base on the ground with its top on the wall, so that the ladder passes over the top of the lamp. Let θ be the angle of elevation of the ladder. If $L(\theta)$ determines the ladder length, determine its critical point. (You may assume this produces the absolute minimum length.)

(**HINT:** After setting $L'(\theta) = 0$, rearrange the equation to get $\tan^3(\theta)$ equaling a constant.)



We split the total ladder length into two separate hypotenuses $L = c_1 + c_2$, where c_1 is the portion of the hypotenuse below $(30\sqrt{3}, 10)$ and c_2 is the portion above $(30\sqrt{3}, 10)$. This yields the following two right triangles:





Using SOHCAHTOA with each, we obtain

$$\sin(\theta) = \frac{10}{c_1} \quad \Rightarrow \quad c_1 = \frac{10}{\sin(\theta)} \qquad\qquad \cos(\theta) = \frac{30\sqrt{3}}{c_2} \quad \Rightarrow \quad c_2 = \frac{30\sqrt{3}}{\cos(\theta)}$$

Therefore, our objective is to minimize the sum $c_1 + c_2$:

$$L(\theta) = \frac{10}{\sin(\theta)} + \frac{30\sqrt{3}}{\cos(\theta)}$$

The domain consists of acute angles: $(0, \pi/2)$.

Taking the derivative, setting it to 0, rearranging and cross-multiplying:

$$L'(\theta) = \frac{-10\cos(\theta)}{\sin^2(\theta)} + \frac{30\sqrt{3}\sin(\theta)}{\cos^2(\theta)} = 0 \quad \Leftrightarrow \quad \frac{10\cos(\theta)}{\sin^2(\theta)} = \frac{30\sqrt{3}\sin(\theta)}{\cos^2(\theta)} \quad \Leftrightarrow \quad 10\cos^3(\theta) = 30\sqrt{3}\sin^3(\theta)$$

Divide $\cos^3(\theta)$ and $30\sqrt{3}$ to turn this into $\tan^3(\theta) = 10/(30\sqrt{3}) = 1/(3\sqrt{3})$. Take cube roots and simplify to get $\tan(\theta) = 1/\sqrt{3}$. Therefore, $\boxed{\theta = \arctan(1/\sqrt{3}) = \pi/6}$.

As directions indicate, you are **not required** to confirm this yields an absolute minimum. (The easiest way would be to check the end behavior: $L(\theta) \to \infty$ as either $\theta \to 0^+$ or $\theta \to \pi/2^-$, so our sole critical point would have to be a minimum.)