Kossack 2015 Answers, by Dr. Michael Klipper

(There may be more approaches than what are shown here.)

1. PROBLEM: *Suppose that* $\lim_{x\to a} f(x) = b$ *and* $\lim_{x\to a} g(x) = c$*. Find* $\lim_{x\to a} f(x)g(x)$.

   Assuming that b and c are real numbers, by the limit laws, this limit is just $\boxed{bc}$.

   (The question would be more ambiguous if we allowed $\pm\infty$ as answers, since $b = \infty$ and $c = 0$ would lead to $\infty \cdot 0$ indeterminate forms...)

2. PROBLEM: *A tangent line is drawn to the hyperbola* $xy = c$ *at the point* $(a, c/a)$*. Show that the area bounded by the coordinate axes and this tangent line is independent of* a*, and find this area.*

   First, to form this tangent line, we find its slope. Since $y = c/x = cx^{-1}$, we get $y' = -cx^{-2} = -c/x^2$. At $x = a$, we get slope $m = -c/a^2$. Therefore, the point-slope form of the tangent line is

   $$\text{Tangent line:} \qquad y - \frac{c}{a} = \frac{-c}{a^2}(x - a)$$

   The x-intercept occurs when $y = 0$, so $-c/a = -c/(a^2) \cdot (x - a)$, then $a = x - a$, so $x = 2a$. The y-intercept occurs when $x = 0$, so $y - c/a = -c/(a^2) \cdot -a = c/a$ and hence $y = 2c/a$.

   If we interpret "area" to mean "signed area" in this problem, the region enclosed by the tangent line and the coordinate axes is a right triangle with base length $2a$ and height $2c/a$, so the area is

   $$\boxed{\text{Area } = \frac{1}{2} \cdot 2a \cdot \frac{2c}{a} = 2c}$$

   However, if we interpret "area" as "total area", then the area is $\boxed{2|c|}$.

3. PROBLEM: *Show that at each point where the curves* $y = ax^3$ *and* $x^2 + 3y^2 = b$ *intersect, their tangent lines are perpendicular.*

   (It is assumed that a, b are constants.) We need to determine the derivatives of both curves to show that they form opposite reciprocals. First, $x^2 + 3y^2 = b$ is easier to differentiate implicitly, leading to

   $$2x + 6y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{6y} = \frac{-x}{3y}$$

   The opposite reciprocal of this slope would be $3y/x$.

Next, we can differentiate $y = ax^3$ explicitly, or we can write it as $y/x^3 = a$ and differentiate implicitly. Taking the first approach, we get $\frac{dy}{dx} = 3ax^2$. To show that this equals $3y/x$ as needed,

$$\frac{3y}{x} = \frac{3ax^3}{x} = 3ax^2 = \frac{dy}{dx}$$

proving the claim. □

4. PROBLEM: *For what values of c does the equation $\ln(x) = cx^2$ have exactly one solution?*

- CASE 1: When $c < 0$, the parabola opens downward, so it is strictly decreasing for $x > 0$. However, $\ln(x)$ is strictly increasing for all $x > 0$. Note how $c(0)^2 = 0$ and $\ln(x) \to -\infty$ as $x \to 0^+$, so $y = cx^2$ starts above $y = \ln(x)$ for $x \approx 0^+$. On the other hand, $c(1)^2 = c < 0$ and $\ln(1) = 0$, so cx^2 lies under $\ln(x)$ at $x = 1$. Because these curves switch which one is bigger on $(0, 1]$, they must cross at some point in $(0, 1)$. Furthermore, because cx^2 keeps falling whereas $\ln(x)$ keeps rising, they can never cross again.

  To make this more precise, however, we make a function to represent the difference between the curves: $f(x) = cx^2 - \ln(x)$. This function is continuous and differentiable for all $x > 0$, and $f(x) = 0$ if and only if $cx^2 = \ln(x)$, so we want to figure out which values of c make it so $f(x)$ has exactly one root. Because $f(x) \to \infty > 0$ as $x \to 0^+$ and $f(1) = c - \ln(1) = c < 0$, the Intermediate Value Theorem says that $f(x)$ must hit zero at some value in $(0, 1)$. Furthermore, it will also help to work out $f'(x)$:

  $$f'(x) = 2cx - \frac{1}{x} = \frac{2cx^2}{x} - \frac{1}{x} = \frac{2cx^2 - 1}{x}$$

  When $c < 0$, this shows $f'(x)$ is always negative, and therefore it never equals 0. By Rolle's Theorem, this means $f(x)$ can never repeat a value (i.e. we can't have $f(a) = f(b)$ for some $a, b > 0$), so there can only be one root of $f(x)$.

- CASE 2: When $c = 0$, clearly $y = 0$ and $y = \ln(x)$ only intersect at $x = 1$, so $c = 0$ also counts for this problem.

- CASE 3: When $c > 0$, we note that $f(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$. Therefore, if $f(x)$ ever attains a negative value for x in $(0, \infty)$, it switches sign at least twice and must cross $f(x) = 0$ at least twice due to the Intermediate Value Theorem. However, if we can show that the absolute minimum of $y = f(x)$ leads to $f(x) = 0$, then that will imply that location is the sole place where $f(x) = 0$.

  Using the formula from $f'(x)$ earlier, but now noting that c is positive, we get $f'(x) = 0$ if and only if the numerator is zero: $2cx^2 - 1 = 0$, so $x^2 = 1/(2c)$

and hence $x = \sqrt{1/(2c)}$. (Remember that the domain of $f(x)$ only has positive numbers, so the negative square root is not relevant.) At this point, we get

$$f\left(\frac{1}{\sqrt{2c}}\right) = c\left(\sqrt{\frac{1}{2c}}\right)^2 - \ln\left(\frac{1}{\sqrt{2c}}\right) = \frac{c}{2c} - \ln\left((2c)^{-1/2}\right) = \frac{1}{2} - \left(\frac{-1}{2}\right)\ln(2c)$$

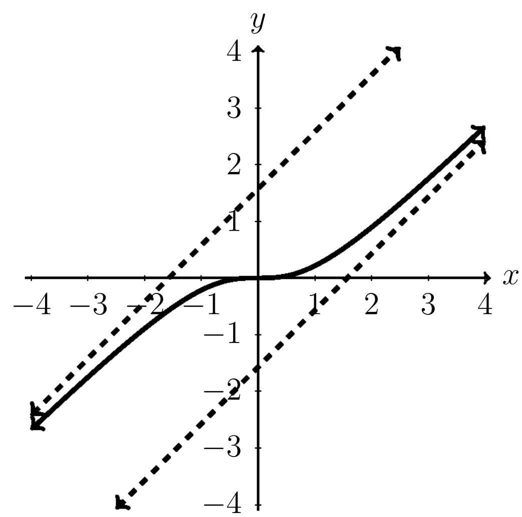
The earlier comments indicate that $f(x) = 0$ exactly once only when this absolute minimum is zero, so $(1/2)+(1/2)\ln(2c) = 0$. This means $\ln(2c) = -1$, so $2c = e^{-1}$, and thus $c = 1/(2e)$.

CONCLUSION: $y = cx^2$ and $y = \ln(x)$ intersect exactly once when

$$\boxed{c \leq 0 \text{ or when } c = 1/(2e)}$$

5. PROBLEM: *Sketch the graph of $y = x - \tan^{-1}(x)$. Show that the graph has 2 slant asymptotes and find them.*

A sketch is provided, done by computer. (I personally wouldn't require people to make a sketch, so I'm alright with using a tool here that wouldn't be allowed on the original exam. If we needed to do this by hand, I'd plot $y(0) = 0$, use the derivative to determine where the graph rises and falls, and I would also try to compute $\lim_{x\to\infty} y = \infty$ and $\lim_{x\to-\infty} y = -\infty$. I wouldn't bother with the second derivative and finding concavities.) The slant asymptotes from the answers at the end are also drawn.



If $y = mx + b$ is a slant asymptote for some constants m and b, it means that either $\lim_{x\to\infty}(x - \tan^{-1}(x)) - (mx + b) = 0$ or that $\lim_{x\to-\infty}(x - \tan^{-1}(x)) - (mx + b) = 0$. (Namely, the distance from the curve to the line approaches 0 over time.)

- First, for the limit $x \to \infty$, we rewrite this limit as

$$\begin{aligned} 0 &= \lim_{x\to\infty}\left(x - \tan^{-1}(x) - mx - b\right) = \lim_{x\to\infty}\left((1-m)x - \tan^{-1}(x) - b\right) \\ 0 &= \left(\lim_{x\to\infty}(1-m)x\right) - (\pi/2) - b \end{aligned}$$

The only time when $(1-m)x$ has a finite limit for $x \to \infty$ is when $1-m=0$, so $m=1$. Therefore, our equation above becomes $0 = -(\pi/2) - b$, so $b = \pi/2$.

- For the limit $x \to -\infty$, the same algebra steps work at first, but we get

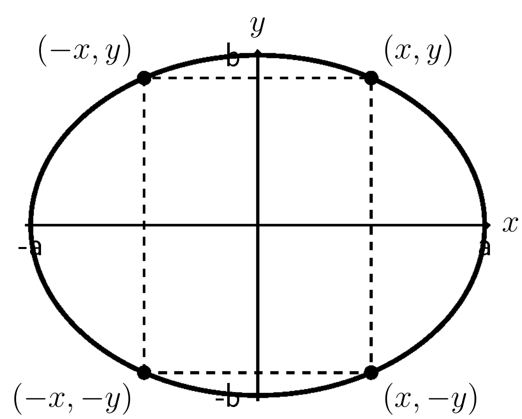
$$0 = \lim_{x\to-\infty} (1-m)x - \lim_{x\to-\infty} \tan^{-1}(x) - b = \left(\lim_{x\to-\infty} (1-m)x\right) - (-\pi/2) - b$$

As before, the remaining limit exists only when $1-m=0$, so $m=1$. We then get $0 = \pi/2 - b$, so $b = -\pi/2$.

CONCLUSION: The slant asymptotes are $y = x + \pi/2$ and $y = x - \pi/2$.

6. PROBLEM: *Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.*

(Suppose a, b are positive constants.) To set up an optimization problem, suppose (x, y) represents the *upper-right* corner of the rectangle we would inscribe in the ellipse. From there, in order to obtain the largest area of a rectangle, our four corners would be located at $(\pm x, \pm y)$. (We are assuming the rectangle of largest area has sides parallel to the coordinate axes, so it has two horizontal sides and two vertical legs.) See this picture:



This rectangle has width $2x$ and height $2y$, so the area is $A = 4xy$. To eliminate a variable, we solve for y in the ellipse (using only the nonnegative solution):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Leftrightarrow \quad \left(\frac{y}{b}\right)^2 = 1 - \frac{x^2}{a^2} \quad \Leftrightarrow \quad y = b \cdot \sqrt{1 - \frac{x^2}{a^2}}$$

Therefore, the area of the rectangle as a function of x is

$$A(x) = 4bx \cdot \sqrt{1 - \frac{x^2}{a^2}}$$

The upper right corner can only use x in $[0, a]$, so that's the domain of our objective function.

Since $A(x)$ is never negative, it's actually easier to optimize the square $f(x) = A(x)^2$ instead:

$$f(x) = A(x)^2 = 16b^2x^2\left(1 - \frac{x^2}{a^2}\right) = 16b^2x^2 - \frac{16b^2}{a^2}x^4 = 16b^2\left(x^2 - \frac{1}{a^2}x^4\right)$$

Now we compute the derivative and factor:

$$f'(x) = 16b^2\left(2x - \frac{1}{a^2}4x^3\right) = 32b^2x\left(1 - \frac{2x^2}{a^2}\right)$$

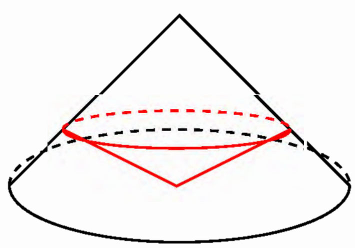
We get $f'(x) = 0$ when $x = 0$ (a domain endpoint) or when $2x^2/a^2 = 1$, so that $x = \sqrt{a^2/2} = a/\sqrt{2}$. At that latter point, it's clear that the derivative switches from positive to negative there, so this means our sole critical value is a local maximum of $f(x)$. With only one local extremum, it means we have found the absolute maximum of $f(x)$.

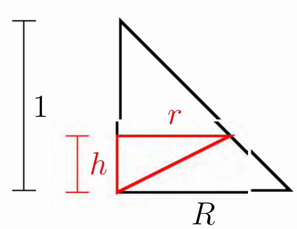
Since $f(x) = A(x)^2$ has its absolute maximum at the same location as $A(x)$, the absolute maximum area is

$$\begin{aligned}\text{Maximum Area} &= A\left(\frac{a}{\sqrt{2}}\right) = 4b \cdot \frac{a}{\sqrt{2}}\sqrt{1 - \frac{(a^2/2)}{a^2}} \\ &= \frac{4}{\sqrt{2}}ab\sqrt{1 - \frac{1}{2}} = \frac{4}{(\sqrt{2})^2}ab = \boxed{2ab}\end{aligned}$$

7. PROBLEM: *A cone is inscribed in a larger cone of height* 1*, so that its vertex is at the center of the larger cone. Show that the inner cone has maximum volume when its height is* 1/3.

CLARIFICATION: When the problem says "vertex is at the center", they must mean the center of the cone's circular opening, not at the center of mass of the cone. Namely, the picture is a bit like the diagram on the left. We also draw a 2D version of half of this picture, sliced through the center axis of the cones, on the right.





In these pictures, R is the (unspecified) radius of the larger cone, and r is the radius of the smaller (red) cone, whereas h is the height of the smaller cone. Our goal is to maximize the volume of the smaller cone, which is

$$V = \frac{\pi}{3}r^2h$$

To eliminate a variable from this formula, think of the hypotenuse of the black triangle as a line through $(0,1)$ and $(R,0)$, so its slope is $-1/R$. The line formula is $y = -x/R+1$, which means that when $x = r$ and $y = h$, we get $h = -r/R+1$. Therefore, we get our objective function

$$V(r) = \frac{\pi}{3}r^2\left(\frac{-r}{R}+1\right) = \frac{\pi}{3}\left(\frac{-1}{R}r^3 + r^2\right)$$

where we treat R like a constant, and the domain of r is $[0,R]$.

To optimize this, compute the derivative

$$V'(r) = \frac{\pi}{3}\left(\frac{-1}{R}3r^2 + 2r\right) = \frac{\pi}{3}\left(\frac{-3r^2+2rR}{R}\right) = \frac{\pi}{3R}r(-3r+2R)$$

Therefore, $V'(r) = 0$ when $r = 0$ or when $-3r+2R = 0$, meaning $r = 2R/3$. A quick check shows that $V(0) = 0$ and that $V(R) = 0$ (because we get $h = 0$ at $r = R$), so the absolute maximum of $V(r)$ occurs at $r = 2R/3$. Thus, the height of the smaller cone is

$$h = \frac{-r}{R}+1 = \frac{-(2R/3)}{R}+1 = \frac{1}{3}$$

as needed. □

8. PROBLEM: *If* $x\sin(\pi x) = \int_0^{x^2} f(t)\,dt$*, where* $f(x)$ *is continuous, find* $f(4)$*.*

First, given that we have an integral of a continuous function with our independent variable in the top limit, let's take the derivative of both sides of this equation and use Part 1 of the Fundamental Theorem on the right side. Think of the right side as a Chain Rule setup of $\int_0^u f(t)\,dt$ with inner $u = x^2$, and we get

$$\frac{d}{dx}\left(\int_0^{x^2} f(t)\,dt\right) = f(x^2)\cdot\frac{d}{dx}(x^2) = 2xf(x^2)$$

Using the Product Rule for the derivative of the left side, we therefore get

$$\sin(\pi x) + x\cos(\pi x)\cdot\pi = 2xf(x^2)$$

Plug in $x = 2$ to get $4f(4) = \sin(2\pi) + 2\pi\cos(2\pi) = 0 + 2\pi(1)$, so $\boxed{f(4) = \pi/2}$.