Topology Qualifying Exam

August 12, 2003
9:00 am - 12:00 noon

1. Prove: If X is compact and $f : X \to Y$ is continuous then $f(X)$ is compact.

2. The *diagonal* of a product space $X \times X$ is the set $\Delta = \{(x, y) \in X \times X \mid x = y\}$. Prove that X is Hausdorff if and only if Δ is a closed subset of $X \times X$.

3. (a) State the definition of homotopy equivalence of topological spaces.
(b) Using this definition, prove that the "theta space" X and the "figure eight space" Y are homotopy equivalent:

$$X = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\} \cup \{(x, y) \in R^2 \mid -1 \leq x \leq 1,\ y = 0\},$$

$$Y = \{(x, y) \in R^2 \mid x^2 + (y-1)^2 = 1\} \cup \{(x, y) \in R^2 \mid x^2 + (y+1)^2 = 1\}.$$

4. Suppose $p : X \to Y$ is a covering map with $p(x_0) = y_0$. Assume that W is path-connected and locally path-connected, and that $f : W \to Y$ is a continuous map with $f(w_0) = y_0$.
(a) Give necessary and sufficient conditions for the existence of a continuous map $g : W \to X$ such that $g(w_0) = x_0$ and $p \circ g = f$. (You do not have to prove your answer.)
(b) Give two examples, one where the conditions are met and one where they fail.

5. Let X be the quotient space obtained from the unit disk D^2 by identifying points on the boundary that are 120° apart. Compute the fundamental group $\pi_1(X)$.

6. Let n be an integer, and suppose the space X is obtained by attaching a 3-ball B^3 to a 2-sphere S^2 by a map of degree n from the boundary of the 3-ball to the 2-sphere. (In other words, given a map $f : S^2 \to S^2$ of degree n, the space X is the quotient space obtained from the disjoint union of B^3 and a S^2 by identifying x with $f(x)$ for all x in the boundary of B^3.) What are the homology groups of X?

7. Compute the homology groups of the space X obtained from the torus $S^1 \times S^1$ by attaching a 2-disk to the circle $S^1 \times \{p\}$ and a second 2-disk to the circle $\{p\} \times S^1$, for p a point of S^1.